Skeletons and superscars

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Abstract

Semiclassical wave functions in billiards based on the Maslov - Fedoriuk approach are constructed. They are defined on classical constructions called skeletons which are the billiards generalization of Arnold's tori. Skeletons in the rational polygon billiards considered in the phase space can be closed with a definite genus or can be open being a cylinder-like or Möbius-like bands. The skeleton formulation is applied to calculate semiclassical wave functions and the corresponding energy spectra for the integrable and pseudointegrable billiards as well as in the limiting forms in some cases of chaotic ones. The superscars of Bogomolny and Schmit are shown to be simply singular semiclassical solutions of the eigenvalue problem in the billiards well built on the singular skeletons in the billiards with flat boundaries in both the integrable and the pseudointegrable billiards as well as in the chaotic cases of such billiards.

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1 Introduction

Billiards while a non-analytic motion area are however well known as examples of the non-integrable two dimensional systems except the known cases of the integrable elliptical, rectangular and some triangle billiards. They are widely considered as a simple field of experimental [1, 12, 3] as well as theoretical [4, 5, 6, 7] (and papers cited there) and computational investigations [8, 9, 10] allowing to apply many different methods (see Sarnak's lecture [11] and [12] of the same author for an extensive review of the respective theoretical methods covering also billiards manifolds as well as the students book of Tabachnikov [13]).

In this paper we are going to develop the semiclassical wave function (SWF) formalism which can be applied to non-integrable cases of the two dimensional motions in billiards and which can be easily extended to higher dimensions.

Essentially our approach is very close to the one of Maslov and Fedoriuk [14]. The main difference between them is in a treatment of crossing the singular points of the SWF's set on caustics. Namely, instead of making the canonical phase space variable transformations accompanied by the Fourier transformations of the SWF's to move through the caustic points we apply the analytical continuation on the complex time plane to both the SWF's and the classical trajectories. This greatly simplifies the corresponding procedure in comparison with the Maslov and Fedoriuk treatment and allows us for not leaving the configuration space [22]. However in this paper there is no opportunity to use the simplification mentioned.

It is further a classical construction which we called **skeletons** on which the SWF's are defined. Each skeleton is compound in a closed way of bundles of rays (classical trajectories). SWF's of basic forms are defined just first on bundles while the **global** SWF's (GSWF) satisfying vanishing boundary conditions of Dirichlet or Neumann are sums of these **basic** SWF's (BSWF) and are uniquely and continuously defined on the skeleton. Skeletons play in this way a role of Arnold's tori [15] except that a number of ray bundles in skeletons can be infinite if billiards with chaotic motions are considered.

When forms of billiards boundaries are considered and their relationships with types of the classical motion in them one realizes that billiards can be dividing into two general classes: the one with the flat boundaries, i.e. the polygon billiards and the second class in which some pieces of their boundaries have finite curvature. In the polygon billiards there are no caustic phenomenon and this is the main and essential property which differs both the classes.

However among the polygon billiards one can still distinguish a class of the rational billiards, i.e. which all angles are rational part of π. As it was shown by Richens and Berry [8] (see also Tabachnikov [13]) the rational billiards can be classified according to classical motions in the corresponding phase space which is performed on a set of two-dimensional disjoint compact Lagrangian surfaces which collecting together can be made equivalent topologically to a compact close surface of a given genus defined by the billiards. All possible values of genus can be realized by the rational billiards.

As an archetype of the rational pseudointegrable polygon billiards can be considered the broken rectangular billiards, i.e. billiards which can be glued of a finite number of rectangles. The SWF's in the rectangular billiards and in the broken ones are just studied intensively in this paper. Nevertheless other simple polygons such as the equilateral triangles and the pentagon billiards are also considered.

It is well known that the semiclassical limits of wave functions are typically singular, i.e. the semiclassical wave functions are frequently deprived of such properties as finiteness and smoothness which have to be satisfied by the exact wave functions. Nevertheless SWF's alone

as well as accompanied them energy spectra give typically very good approximations to the exact ones in the high energy limit.

However it happens frequently that semiclassical calculations provide us with exact wave functions and energy spectra. This can takes place for example when the calculated SWF's satisfy the same conditions as the exact ones and the classically allowed configuration space coincides with the quantum one. Such SWF's are called **regular** in this paper. Examples of such regular semiclassical solutions are provided by the generic SWF in the rectangle, the equilateral triangle and the pentagon billiards.

Most of the semiclassical wave functions which solve the eigenvalue problem in the semiclassical limit are however **singular**, i.e. they do not satisfy some of the conditions which the exact solutions have to do. The SWF's which can be constructed for the rectangular billiards and the broken rectangle ones are not exceptional in the respects mentioned and can be also of two types - the regular and the singular ones depending on skeletons used to their constructions. It is just the singular SWF's for which superscars phenomenon of Bogomolny *et al* [17, 18] can be observed. While this singular behaviour of SWF's in the cases considered is reduced only to discontinuities of the first derivatives of the respective SWF's this "defect" of them is enough for being a source of the superscar phenomenon.

It is also shown in the paper that the superscars are typical not only for the integrable and pseudointegrable systems but also for the chaotic ones. A well known example is the Bunimovich stadium with its bouncing ball modes [3, 4, 5]. But it is easy to give many other examples of the chaotic billiards with any form of the superscars which can be found in the broken rectangular billiards.

The paper is organized as follows.

In the next section the Maslov - Fedoriuk method of the semiclassical wave function construction is reminded and discussed.

In sec.3 a construction of skeletons is given.

In sec.4 global SWF's in billiards are constructed.

In sec.5 the rectangular billiards, the broken rectangle ones, the triangle and the pentagon billiards are considered. In the rectangular billiards case all possible SWF's which can be defined in it, i.e. the regular and the singular ones, are discussed. SWF's in the remaining billiards are considered selectively because of increasing complexity of the corresponding skeletons. Nevertheless it is shown that in these cases the superscar SWF's are common and more spectacular than for the rectangular billiards.

Next in sec.6 it is shown that the superscar SWF's can be implemented into chaotic deformations of the broken rectangle, the triangle and the pentagon billiards.

In sec.7 the results of the paper are summarized.

There are two appendixes attached to the paper which justify the main assumptions used in the construction of the global SWF's on skeletons.

2 Semiclassical wave function expansion for n-D stationary Schrödinger equation

Consider the n-dimensional stationary Schrödinger equation:

$$\triangle\Psi(\mathbf{r}) + \lambda^2\frac{2m}{\hbar^2}(E - V(\mathbf{r}))\Psi(\mathbf{r}) = 0 \qquad (1)$$

with a potential $V(\mathbf{r})$, $\mathbf{r} \in R_n$ confining a point particle with a mass m and containing a formal dimensionless parameter $\lambda > 0$. For a convenience we shall put further $\hbar = 1$ and $m = 1$. The Schrödinger equation is recovered by putting $\lambda = 1$ in (1).

We would like to construct a solution to Eq.(1) using the idea of Maslov *et al* [14] and considering the wave function $\Psi(\mathbf{r})$ as defined on families of classical trajectories a dynamic of which is given by the classical Hamiltonian $H = \frac{1}{2}\mathbf{p}^2 + V(\mathbf{r})$ and which carry an energy E_0 all.

Such families are constructed locally in the following way.

In R_n we choose a $n-1$-D hypersurface Σ_{n-1} parametrized by local coordinates $(s_1, ..., s_{n-1})$. On Σ_{n-1} the initial momenta $\mathbf{p}(\mathbf{r}_0)$, $\mathbf{r}_0 \in \Sigma_{n-1}$, are defined so that the pair $(\mathbf{r}_0, \mathbf{p}(\mathbf{r}_0))$, $\mathbf{r}_0 \in \Sigma_{n-1}$ serve as the initial data for the trajectories $\mathbf{r}(t) = \mathbf{f}(\mathbf{r}_0, \mathbf{p}(\mathbf{r}_0); t)$ developed by the Hamiltonian H. Additionally the momentum field $\mathbf{p}(\mathbf{r}_0)$ defined on Σ_{n-1} has to satisfy:

$$\oint_C \mathbf{p}(\mathbf{r}_0) d\mathbf{r}_0 = 0 \tag{2}$$

for each loop C, $C \subset \Sigma_{n-1}$.

We can now define the transformation: $\mathbf{r} \to (t, \mathbf{r}_0) \to (t, s_1, ..., s_{n-1})$, $\mathbf{r} \equiv (x_1, ..., x_n)$, $\mathbf{r}_0 \equiv (x_{0,1}(s_1, ..., s_{n-1}), ..., x_{0,n}(s_1, ..., s_{n-1}))$, which is one-to-one up to a caustic surface C_{n-1} on which the Jacobean $(\mathbf{f}(\mathbf{r}_0, \mathbf{p}(\mathbf{r}_0); t) \equiv \bar{\mathbf{f}}(t, s_1, ..., s_{n-1}))$:

$$J(t, s_1, ..., s_{n-1}) = \left| \frac{\partial \bar{f}_i}{\partial t}, \frac{\partial \bar{f}_i}{\partial s_j} \right| \tag{3}$$

vanishes.

A n-dimensional domain Λ_n of $2n$-dimensional phase space R_{2n} made in this way by the hypersurface Σ_{n-1} and trajectories emerging from it is known as the Lagrange manifold [15].

Therefore in the variables $t, s_1, ..., s_{n-1}$ the new wave function $\chi(t, s_1, ..., s_{n-1})$ satisfies the following relation with the previous one:

$$|\chi(t, s_1, ..., s_{n-1})|^2 = |\Psi(\bar{\mathbf{f}}(t, s_1, ..., s_{n-1})|^2 |J(t, s_1, ..., s_{n-1})| \tag{4}$$

The particle momentum $\mathbf{p}$ on the trajectories $\mathbf{r}(t) = \bar{\mathbf{f}}(t, s_1, ..., s_{n-1})$ satisfies of course the equation:

$$\frac{\partial \bar{\mathbf{f}}(t, s_1, ..., s_{n-1})}{\partial t} = \mathbf{p}(\bar{\mathbf{f}}(t, s_1, ..., s_{n-1})) \tag{5}$$

defining also the Jacobean evolution. Namely:

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial \bar{f}_i(t, s_1, ..., s_{n-1})}{\partial t} &= \sum_{j=1}^{n} \frac{\partial p_i}{\partial x_j} \frac{\partial \bar{f}_j(t, s_1, ..., s_{n-1})}{\partial t} \\ \frac{\partial}{\partial t} \frac{\partial \bar{f}_i(t, s_1, ..., s_{n-1})}{\partial s_l} &= \sum_{j=1}^{n} \frac{\partial p_i}{\partial x_j} \frac{\partial \bar{f}_j(t, s_1, ..., s_{n-1})}{\partial s_l} \\ l &= 1, ..., n-1 \end{aligned} \tag{6}$$

so that

$$\frac{\partial J(t, s_1, ..., s_{n-1})}{\partial t} = J(t, s_1, ..., s_{n-1}) \nabla \mathbf{p}(\bar{\mathbf{f}}(t, s_1, ..., s_{n-1})) \tag{7}$$

The above equation is just the Liouville theorem with the solution:

$$J(t,s_1,...,s_{n-1})=J(s_1,...,s_{n-1})e^{\int_0^t \nabla \mathbf{p}(\bar{\mathbf{f}}(t',s_1,...,s_{n-1}))dt'} \tag{8}$$

where $J(s_1,...,s_{n-1})$ is the value of the Jacobean on the hypersurface Σ_{n-1}.

It is well known from the classical Hamiltonian mechanics [15] that the action integral:

$$S(\mathbf{r},\mathbf{r}_0)=\int_{\mathbf{r}_0}^{\mathbf{r}}\mathbf{p}(\mathbf{r}')d\mathbf{r}' \tag{9}$$

taken on the Lagrange manifold Λ_n is a point function of $\mathbf{r}$ and $\mathbf{r}_0$. Therefore taking $\mathbf{r}_0$ as a definite fixed point of the hypersurface Σ_{n-1} and denoting by $S(\mathbf{r})$ the action function corresponding to this case we can complete a definition of the wave function $\chi(t,s_1,...,s_{n-1})$ by the following equation:

$$\Psi^\sigma(\bar{\mathbf{f}}(t,s_1,...,s_{n-1}))=J^{-\frac{1}{2}}(t,s_1,...,s_{n-1}))e^{\sigma\lambda iS(\bar{\mathbf{f}}(t,s_1,...,s_{n-1})}\chi^\sigma(t,s_1,...,s_{n-1}) \tag{10}$$

where $\sigma=\pm$ is a signature of $\Psi^\sigma(\mathbf{r})$.

The form (10) of the semiclassical wave functions (SWF) will be called **basic** (BSWF).

Therefore the quantities involved in the above definitions satisfy the following equations:

$$\begin{aligned}
\mathbf{p}(\mathbf{r})&=\nabla S(\mathbf{r})\\
\frac{1}{2}\mathbf{p}^2(\mathbf{r})+V(\mathbf{r})-E_0&=0\\
\triangle(J^{-\frac{1}{2}}\chi^\sigma(\mathbf{r}))+\sigma 2i\lambda J^{-\frac{1}{2}}(\mathbf{r})\nabla\chi^\sigma(\mathbf{r})\cdot\mathbf{p}(\mathbf{r})+2\lambda^2(E-E_0)J^{-\frac{1}{2}}(\mathbf{r})\chi^\sigma(\mathbf{r})&=0\\
\mathbf{r}&=\bar{\mathbf{f}}(t,s_1,...,s_{n-1})
\end{aligned} \tag{11}$$

By the variables $t,s_1,...,s_{n-1}$ the third of the last equations can be rewritten in the following form:

$$\begin{aligned}
\sigma 2i\lambda\frac{\partial\chi^\sigma(t,s_1,...,s_{n-1},\lambda)}{\partial t}&+\\
J^{\frac{1}{2}}\triangle\left(J^{-\frac{1}{2}}\chi^\sigma(t,s_1,...,s_{n-1},\lambda)\right)+\lambda^2(E-E_0)\chi^\sigma(t,s_1,...,s_{n-1},\lambda)&=0
\end{aligned} \tag{12}$$

where a dependence of $\chi^\sigma(t,s_1,...,s_{n-1},\lambda)$ on λ was shown explicitly.

The Eq.(12) describes the time evolution of $\chi^\sigma(t,s_1,...,s_{n-1},\lambda)$ along trajectories starting on the hypersurface Σ_{n-1} if its "initial" values on this surface, i.e. $\chi^\sigma(0,s_1,...,s_{n-1},\lambda)\equiv\chi^\sigma(s_1,...,s_{n-1},\lambda)$ are given.

We are going to consider the equation (12) in the semiclassical limit $\lambda\to+\infty$ looking for its solutions in the form of the following asymptotic series:

$$\begin{aligned}
E-E_0&=\sum_{k\geq 1}E_k\lambda^{-k-1}\\
\chi^\sigma(t,s_1,...,s_{n-1},\lambda)&=\sum_{k\geq 0}\chi_k^\sigma(t,s_1,...,s_{n-1})\lambda^{-k}\\
\chi^\sigma(s_1,...,s_{n-1},\lambda)&=\sum_{k\geq 0}\chi_k^\sigma(s_1,...,s_{n-1})\lambda^{-k}
\end{aligned} \tag{13}$$

Putting $\lambda=1$ in (1), (10) and (13) we get approximate semiclassical solutions to the energy eigenvalue problem of the Schrödinger equation.

It is to be noticed that for the selfconsistency reasons the semiclassical series for the energy parameter in (13) starts from the second power of λ^{-1}, i.e. this ensures the proper hierarchy of steps in the algorithm of semiclassical calculations by which the higher order terms of the series in (13) are determined by the lower order ones.

It should be noticed also that despite the fact that E_0 enters the classical equation of motion (11) it is still quantum, i.e. its value depends on $\hbar$ which is considered to have the definite numerical value, i.e. $\hbar$ is not a parameter. In particular the series (13) represent the inverse power hierarchy in the formal parameter λ, i.e. not in powers of $\hbar$, between subsequent terms.

Moreover E_0 if quantized can depend on λ. However, whatever this dependence is the semiclassical series of the difference $E - E_0$ must be given by (13).

Needless to say the introducing λ makes a treatment of the Schrödinger equation equivalent of course to considering it in the limit $\hbar \to 0$, i.e. semiclassically, clearly however separating the role of $\hbar$ as a parameter from its role defining the microscale of quantum phenomena.

Substituting (13) into (12) we get:

$$\begin{aligned} \frac{\partial \chi_0^\sigma(t, s_1, ..., s_{n-1})}{\partial t} &= 0 \\ \frac{\partial \chi_{k+1}^\sigma(t, s_1, ..., s_{n-1})}{\partial t} &= \\ \frac{\sigma i}{2}\left(J^{\frac{1}{2}} \triangle \left(J^{-\frac{1}{2}}\chi_k^\sigma(t, s_1, ..., s_{n-1})\right) + 2\sum_{l=0}^{k} E_{k-l+1}\chi_l^\sigma(t, s_1, ..., s_{n-1})\right) & \\ k = 0, 1, 2, ..., & \end{aligned} \tag{14}$$

with the obvious solutions:

$$\begin{aligned} \chi_0^\sigma(t, s_1, ..., s_{n-1}) &\equiv \chi_0^\sigma(s_1, ..., s_{n-1}) \\ \chi_{k+1}^\sigma(t, s_1, ..., s_{n-1}) &= \chi_{k+1}^\sigma(s_1, ..., s_{n-1}) + \\ \frac{\sigma i}{2}\int_0^t \left(J^{\frac{1}{2}} \triangle \left(J^{-\frac{1}{2}}\chi_k^\sigma(t', s_1, ..., s_{n-1})\right) + 2\sum_{l=0}^{k} E_{k-l+1}\chi_l^\sigma(t', s_1, ..., s_{n-1})\right) dt' & \\ k = 0, 1, 2, ..., & \end{aligned} \tag{15}$$

For future applications it is worth to note that if (12) is obviously invariant on a reparametrization of the hypersurface Σ_{n-1} it is also invariant on the following change of variables:

$$\begin{aligned} t &\to \tau(s_1, ..., s_{n-1}) \pm t \\ s_k &\to h_k(s_1, ..., s_{n-1}) \\ k &= 1, ..., n-1 \end{aligned} \tag{16}$$

if it is accompanied simultaneously by the transformations:

$$\begin{aligned} \chi^\sigma(t, s_1, ..., s_{n-1}, \lambda) &\to \\ \chi^{\pm\sigma}(t, s_1, ..., s_{n-1}, \lambda) \equiv (\pm J_1)^{-\frac{1}{2}}(h_1^{-1}(s_1, ..., s_{n-1}), ..., h_{n-1}^{-1}(s_1, ..., s_{n-1})) &\times \\ \chi^\sigma(\tau(s_1, ..., s_{n-1}) \pm t, h_1^{-1}(s_1, ..., s_{n-1}), ..., h_{n-1}^{-1}(s_1, ..., s_{n-1}), \lambda) & \end{aligned} \tag{17}$$

where $J_1(s_1, ..., s_{n-1})$ is the Jacobean of the transformation $s_k \to h_k(s_1, ..., s_{n-1})$, $k = 1, ..., n-1$.

3 Skeletons - classical constructions in billiards

Before applying the above formalism to construct continuous semiclassical wave functions inside billiards B (Fig.1) vanishing on its boundary i.e. satisfying the Dirichlet boundary conditions it is first necessary to perform a classical construction consisting of classical trajectories on which the desired SWF's can be defined. This is just **a skeleton** i.e. a closed set of families of trajectories which forms a base on which SWF's can be constructed.

The skeleton idea relies on an observation that the short wave packets propagate in billiards approximately along the straight lines (rays of the geometrical optics) gradually however becoming wider and wider due to unavoidable diffractive effects. Because of the last effects even an initially narrow bundle of such rays fills completely a volume admitted by the respective boundary conditions.

Therefore from the one hand the SWF propagates along the classical objects - trajectories, but from the other hand it has to be defined on a bundle of such trajectories sufficiently wide to gather diffractive effects satisfying nevertheless the rules of the geometrical optics. In particular an effect of shadow typical for the geometrical optics should be observed on a way of propagation of SWF's.

Finally a set of all such bundles should be closed and mutually connected to permit us performing a construction of the global semiclassical wave function from its pieces defined on the separate bundles.

Below a notion of a skeleton is defined which according to our expectations takes into account all the aspects of the semiclassical limit discussed above.

3.1 Ray bundles and bundle skeletons

For the needs of this paper we shall assume that the billiards B is classical and according to Fig.1 it has no holes inside and its boundary ∂B is a closed curve independent of λ and given by $\mathbf{r} = \mathbf{r}_0(s) = [x_0(s), y_0(s)]$ where s is a distance of a boundary point $\mathbf{r}_0(s)$ measured clockwise along ∂B from some other point A of ∂B chosen arbitrary, i.e. $s(A) = 0$. Both $x_0(s)$ and $y_0(s)$ are continues. The curve however consists of a finite number $q,\ q \geq 1$ of smooth arcs or segments $A_1, ..., A_q$ with respective length $L_1, ..., L_q$, so that the derivatives $x_0'(s)$ and $y_0'(s)$ are discontinuous in a finite number of points on the segment $0 \leq s \leq L$ where $L = L_1 + \cdots + L_q$ is the global length of ∂B. Both $x_0(s)$ and $y_0(s)$ are of course periodic with the period equal to L. We shall identify the point A with the point beginning the arc (segment) A_1.

Next we define a bundle of rays as a family of trajectories in the following way.

Let $A_k(u, l),\ L_1 + \cdots + L_{k-1} \leq u \leq L_1 + \cdots + L_k,\ 0 < l \leq L_k$, be an open connected piece of the arc (segment) A_k beginning at $s = u$ and having a length l.

Let further $\mathbf{r}_k(s, t; u, l),\ u < s < u + l,\ 0 \leq t$, be a family of trajectories given by angles $\gamma_k(s; u, l),\ 0 \leq \gamma_k(s; u, l) \leq 2\pi$, at which the trajectories escape from $A_k(u, l)$. The angles $\gamma(s; u, l)$ are smooth functions of s and are measured with respect to the x-axis while the tangential vectors $\mathbf{t}(s) = [\frac{dx_0(s)}{ds}, \frac{dy_0(s)}{ds}] = [\cos\beta(s), \sin\beta(s)]$ are inclined to the x-axis at angles $\beta(s)$ (Fig.1). The latter angle can be discontinuous at the points where $x_0'(s)$ and $y_0'(s)$ are discontinuous. Then the angle $\alpha_k(s; u, l) = \gamma_k(s; u, l) - \beta_k(s)$ is made by the classical ball momentum $\mathbf{p}(s; u, l)$ on the trajectory with the tangent vector $\mathbf{t}(s)$, i.e. $\mathbf{p}(s; u, l) \cdot \mathbf{t}(s) = p \cos\alpha_k(s; u, l)$. It is assumed that $0 < \alpha_k(s; u, l) < \pi$.

The classical time evolution of the family $\mathbf{r}_k(s, t; u, l),\ u < s < u + l$, is therefore the

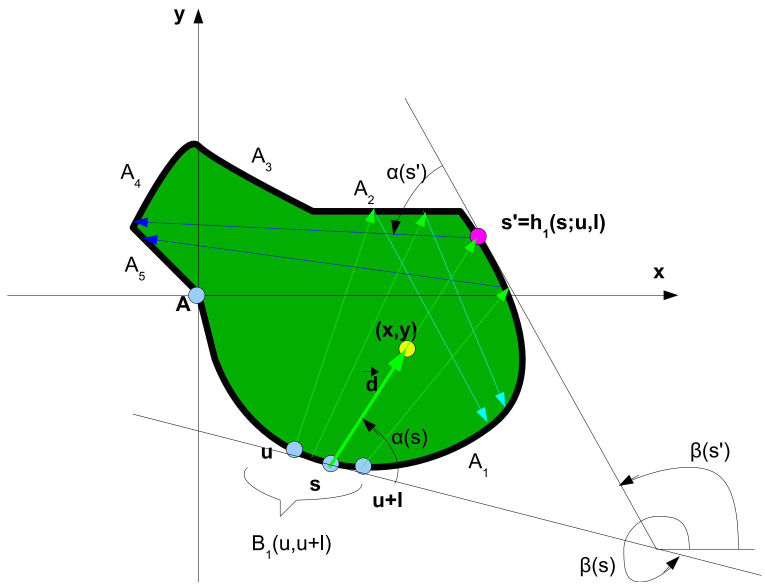


Figure 1: An arbitrary billiards

following

$$\mathbf{r}_k(s,t;u,l) = \mathbf{r}_0(s) + \mathbf{p}(s;u,l)t, \qquad \mathbf{r}_0(s) \in A_k \tag{18}$$

where $\mathbf{p}(s;u,l) = [p\cos\gamma_k(s;u,l), p\sin\gamma_k(s;u,l)]$ satisfies the classical equations of motions (11), i.e. $\mathbf{p}^2(s;u,l) = 2E_0$ (again we put $m = 1$ for the billiard ball mass).

The trajectories (18) define of course the change of variables $(x,y) \to (t,s)$, in vicinity of $A_k(u,l)$, i.e. $x = f_k(t,s;u,l)$, $y = g_k(t,s;u,l)$ with the Jacobean:

$$\tilde{J}_k(t,s;u,l) = p^2\gamma'_k(s;u,l)t - p\,|\mathbf{t}(s)|\sin\alpha_k(s;u,l) = p^2\gamma'_k(s;u,l)t - p\sin\alpha_k(s;u,l) \tag{19}$$

since $|\mathbf{t}(s)| = 1$.

The family of trajectories defined in the above way will be called a **bundle of rays** emerging from the segment $A_k(u,l)$ of A_k and will be denoted by $B_k(u,l)$ while the trajectories themselves will be called **rays**.

Note that each bundle by definition is an open set of rays with at most two limiting rays as its boundary. If the limiting rays coincide then necessarily the corresponding segment $A_k(u,l)$ can be closed to the whole billiards boundary. This is the case for example of a bundle defined in the circle billiards which rays have the same angular momenta each.

Suppose the limiting rays of bundles $B_k(u,l)$ and $B_{k'}(u',l')$ coincide on a piece P of them. Then by closing the bundles on this common boundary we get a bundle which we call **compound** and denote by $B_k(u,l) \cup_P \; B_{k'}(u',l')$. Such a closing operation will be called a **composition** of the initial bundles. Of course each bundle can be decomposed into two others by the reverse operation becoming the composition of the resulting bundles.

A compound bundle can be composed of course from many bundles and its rays emerge then from the sum $\bigcup_k A_k(u,l)$ of the segments of the composing bundles. If $\bigcup_k A_k(u,l)$ is a

connected piece of the billiards boundary then such a composition and its result a compound bundle will be called **regular**. Other cases of compositions and compound bundles will be called **singular**.

Since each ray of the bundle $B_k(u,l)$ after some time $\tau_k(s;u,l)$, $\mathbf{r}_0(s) \in A_k(u,l)$, (different for different rays) achieves another point of the boundary ∂B it means that the bundle $B_k(u,l)$ maps the segment $A_k(u,l)$ into another piece $BA_k(u,l)$ of the boundary ∂B. In general this map of $A_k(u,l)$ into $BA_k(u,l)$ provided by the transformation (18) is one-to-one except the caustic points of $BA_k(u,l)$ in which $\tilde{J}_K(\tau_k(s;u,l),h_k(s;u,l);u,l) = 0$, $\mathbf{r}_0(h_k(s;u,l)) \in BA_k(u,l)$. Here $h_k(s;u,l)$, $\mathbf{r}_0(s) \in A_k(u,l)$, realizes explicitly this map. If however $\frac{\partial h_k(s;u,l)}{\partial s} \neq 0$, $\mathbf{r}_0(s) \in A_k(u,l)$, i.e. this map is one-to-one then such a bundle will be called regular at the boundary ∂B or simply regular.

All bundles considered below will be assumed to be regular.

By $DB_k(u,l)$ will be denoted a domain of the billiards B covered by rays of the bundle $B_k(u,l)$ emerging from $A_k(u,l)$ and ending at $BA_k(u,l)$. The domain $DB_k(u,l)$ is locally a Lagrangian manifold on which each loop integral $\oint \mathbf{p} \cdot d\mathbf{r}$ vanishes.

While a definition of a bundle is essentially local it can happen that in particular cases of bundles or compound bundles they can cover the whole billiards except necessarily some pieces of its boundary, i.e. $\overline{DB_k(u,l)} = B = \overline{D(B_k(u,l) \cup_P B_{k'}(u',l'))}$. Such bundles or compound bundles will be then called **global**.

Assume further the boundary ∂B to be a mirror-like, i.e. reflecting the incoming rays according to the reflection principle of the geometrical optics and let $A_{k'}(u',l')$ be a piece of another arc $A_{k'}$ of ∂B such that $A_{k'}(u',l') \cap BA_k(u,l) \neq \oslash$ on which another regular bundle of rays $B_{k'}(u',l') = \{\mathbf{r}_{k'}(s,t;u',l') : u' < s < u'+l',\ 0 \leq t\}$ is defined.

If on the segment $A_{k'}(u',l') \cap BA_k(u,l)$ the ray bundle $B_{k'}(u',l')$ coincides with the reflected one we call the ray bundle $B_{k'}(u',l')$ a reflection of the bundle $B_k(u,l)$ on the segment mentioned.

The reflection operation over the bundle $B_k(u,l)$ will be denoted by Π so that $\Pi B_k(u,l)$ denotes the set of all rays arising by the reflection of the bundle $B_k(u,l)$ of ∂B.

Consider now a family of the disjoint ray bundles $\mathbf{B} = \bigcup B_k(u,l)$, $B_k(u,l) \cap B_{k'}(u',l') = \oslash$ if $B_k(u,l) \neq B_{k'}(u',l')$.

The family $\mathbf{B}$ will be called **closed** under reflection Π on the boundary ∂B if the following two conditions are satisfied for each bundle $B_k(u,l) \in \mathbf{B}$:

$$\Pi B_k(u,l) = \bigcup_{j=1}^{n} B_j(u_j,l_j) \cap \Pi B_k(u,l), \qquad B_j(u_j,l_j) \in A_{i_j},\ j = 1,...,n$$
$$B_k(u,l) = \bigcup_{i=1}^{m} \Pi B_i(u_i,l_i) \cap B_k(u,l), \qquad B_i(u_i,l_i) \in A_{j_i},\ i = 1,...,m$$
$$\Pi\mathbf{B} = \mathbf{B} \tag{20}$$

A closed family $\mathbf{B}' = \bigcup B_{k'}(u',l')$ is embedded into a closed family $\mathbf{B} = \bigcup B_k(u,l)$ if each ray bundle of $\mathbf{B}'$ is a subset of some ray bundle of $\mathbf{B}$ and each bundle of $\mathbf{B}$ contains some bundle of $\mathbf{B}'$.

A closed bundle family is called connected if a unique possibility to represent it by a sum of another two disjoint closed bundle families is a decomposition operation done on every bundle of the family.

A closed connected bundle family will be called a **Lagrange bundle skeleton** or simply a **skeleton** if it cannot be embedded into another closed connected bundle family.

From now on all considered bundle families will be assumed to be skeletons.

Let us stress the following four basic properties of skeletons which are of great importance:

1. each skeleton is complete i.e. none additional bundle can be added to it not destroying it connectedness,

2. each skeleton cannot be decomposed into "smaller" ones (again by the connectedness property),

3. each bundle of a given skeleton is a result of reflections of other bundles of the same skeleton (by (20)),

4. each ray belonging to a skeleton $\mathbf{B}$ will never leave $\mathbf{B}$ by its time evolution and bounces on the billiards boundary.

Let us now reverse in time all trajectories belonging to $\mathbf{B}$. This operation leads us again to some skeleton $\mathbf{B}^A$ which will be called **associated** with $\mathbf{B}$.

Bundles of $\mathbf{B}^A$ are obtained simply from the corresponding bundles of $\mathbf{B}$. Namely with each bundle $B_k(u,l)$ of $\mathbf{B}$ let us associate a bundle $B_k^A(u,l)$ which trajectories satisfy the following condition:

$$\gamma_k^A(s;u,l) = \pi + 2\beta(s) - \gamma_k(s;u,l), \quad u < s < u + l \tag{21}$$

i.e. these trajectories are just the reflections on ∂B of the time reversed trajectories defined by $\gamma_k(s;u,l)$ and belonging to $B_k(u,l)$.

The skeleton $\mathbf{B}^A$ is organized by all bundles $B_k^A(u,l)$ obtained in the above way.

Needless to say $(\mathbf{B}^A)^A \equiv \mathbf{B}$.

Finally let us denote by $D_{\mathbf{B}} \subset B$ a topological sum of all domains $DB_k(u,l)$, i.e. $D_{\mathbf{B}} = \bigcup_{B_k(u,l)\subset\mathbf{B}} DB_k(u,l)$. Define $D_{\mathbf{B}^A}$ analogously. By the construction of the skeleton $\mathbf{B}^A$ we have $D_{\mathbf{B}^A} \equiv D_{\mathbf{B}}$.

An useful operation on a skeleton is its reduction which means making all possible compound bundles of the bundles of the skeleton. Such a form of the skeleton will be called a **reduced skeleton** and denoted by $\mathbf{B}^R$.

Reduced skeletons although useful are deprived however of many properties of the skeletons. In particular it is typically not possible to construct a skeleton associated with the reduced one. On the other hand in the polygon billiards each compound bundle is associated with a definite momentum of the billiards ball so that to different momenta of the ball correspond different compound bundles.

If a reduced skeleton $\mathbf{B}^R$ contains only global bundles then the skeleton $\mathbf{B}$ will be called **global** and if its bundles are all global and regular the skeleton $\mathbf{B}$ will be called **regular**. Skeletons which are not regular will be called **singular**. It then follows that skeletons can be global but singular.

3.2 Skeletons in the phase space. Pseudointegrable billiards

In the case of the polygon billiards skeletons considered in the phase space are decomposed into separate pieces parallel to the billiards plane. Each a piece corresponds to a separate compound bundle. Any trajectory of the skeleton visits all pieces (bundles) jumping from one piece (bundle) to another when achieving the piece boundary which projected on the billiards plane coincides with the corresponding piece of the billiards boundary.

One can imagine a continues surface made of the pieces mentioned gluing their respective boundaries. Namely, we can get it by gluing points of a piece boundary by which trajectories leave the piece to visit the next one with these entry points of the next piece. If a number of bundles is finite such a construction leads us to a compact two dimensional surface. If such a surface is closed it has then a definite genus. Among the others these are the cases of the global skeletons in the rational polygon billiards [8, 13, 23].

It will appear in sec.5 that the skeletons in the investigated billiards can form Lagrange surfaces in the phase space which are closed according to the above construction with a definite genus but also which are open being a cylinder-like or a Möbius-like bands.

It will appear also in the next sections that only the skeletons which are regular can provide us with GSWF's which are exact solutions of the eigenvalue problems. The singular skeletons provides us with singular GSWF's which cannot be exact and some of them show typical properties of the superscar solutions.

Billiards for which **every** possible skeleton has a finite number of bundles are certainly distinguished by a possibility of construction of all GSWF's for such a billiards in compact finite forms. Therefore it would be reasonable to extend a notion of pseudointegrability introduced by Richens and Berry [8] to such a billiards despite the fact that not all of the skeletons corresponding to them can be global so that the continuous Lagrange surfaces which construction has been described above can be open, i.e. with boundaries.

4 SWF's defined on a skeleton

4.1 BSWF's defined on a bundle

Consider a skeleton $\mathbf{B}$. On each of its ray bundle $B_k(u,l)$ we can now define the following pair of BSWF's $\Psi_k^\sigma(t,s;u,l;\lambda),\ \sigma=\pm$:

$$\Psi_k^\sigma(t,s;u,l;\lambda)=\tilde{J}_k^{-\frac{1}{2}}(t,s;u,l)e^{\sigma i\lambda\left(p^2t+p\int_u^s\cos\alpha_k(s;u,l)ds'\right)}\chi_k^\sigma(t,s;u,l;\lambda) \tag{22}$$

where $p^2=2E_0$ and $\chi_k^\sigma(t,s;u,l;\lambda),\ \sigma=\pm$, are given by (13) and (15).

Exactly in the same way we can define a pair $\Psi_{A;k}^\sigma(t,s;u,l;\lambda),\ \sigma=\pm$, of BSWF's on the corresponding associated bundle $B_k^A(u,l)$:

$$\Psi_{A;k}^\sigma(t,s;u,l;\lambda)=\tilde{J}_{A;k}^{-\frac{1}{2}}(t,s;u,l)e^{\sigma i\lambda\left(p^2t-p\int_u^s\cos\alpha_k(s;u,l)ds'\right)}\chi_{A;k}^\sigma(t,s;u,l;\lambda) \tag{23}$$

It will be also convenient for further considerations to substitute the time variable t by the distance variable $d=pt$ and consequently to give the trajectories (18), the Jacobean (19) and the wave function (22) the following forms:

$$\begin{aligned}\mathbf{r}_k(d,s;u,l)&=\mathbf{r}_0(s)+\mathbf{d}(s;u,l)\\ \mathbf{d}(s;u,l)=\mathbf{p}t&=[d\cos\gamma_k(s;u,l),d\sin\gamma_k(s;u,l)]\\ \mathbf{r}_0(s)&\in A_k(u,l)\end{aligned} \tag{24}$$

and

$$\begin{aligned}J_k(d,s;u,l)=\frac{1}{p}\tilde{J}_k(t,s;u,l)=\frac{\partial\gamma_k(s;u,l)}{\partial s}d-\sin\alpha_k(s;u,l)\\ \mathbf{r}_0(s)\in A_k(u,l)\end{aligned} \tag{25}$$

and

$$\Psi_k^\sigma(d,s;u,l;\lambda) = J_k^{-\frac{1}{2}}(d,s;u,l)e^{\sigma i\lambda p\left(d+\int_u^s \cos\alpha_k(s';u,l)ds'\right)}\bar{\chi}_k^\sigma(d,s;u,l;\lambda)$$
$$\mathbf{r}_0(s) \in A_k(u,l) \qquad (26)$$

where $\bar{\chi}_k^\sigma(d,s;u,l;\lambda) \equiv p\chi_k^\sigma(\frac{d}{p},s;u,l;\lambda)$ and $\sigma = \pm$.

Nevertheless, for simplicity of notations, the bar over $\bar{\chi}_k^\sigma(d,s;u,l;\lambda)$ will be dropped in our further considerations.

By the variable d the solutions (15) can be rewritten in the form:

$$\chi_{k,0}^\sigma(d,s;u,l) \equiv \chi_{k,0}^\sigma(s;u,l)$$
$$\chi_{k,j+1}^\sigma(d,s;u,l) = \chi_{k,j+1}^\sigma(s;u,l) +$$
$$\frac{\sigma i}{2p}\int_0^d \left(\tilde{\triangle}_k(a,s;u,l)\chi_{k,j}^\sigma(a,s;u,l) + 2\sum_{m=0}^{j} E_{j-m+1}\chi_{k,m}^\sigma(a,s;u,l)\right) da$$
$$j = 0,1,2,..., \qquad (27)$$

where $\tilde{\triangle}_k(d,s;u,l) = J_k^{\frac{1}{2}}(d,s;u,l)\cdot\triangle_k(d,s;u,l)\cdot J_k^{-\frac{1}{2}}(d,s;u,l)$ and $\triangle_k(d,s;u,l)$ is the Laplacean expressed by the variables d and s corresponding to the -bundle.

$\Psi_k^\sigma(d,s;u,l;\lambda),\ \sigma = \pm$, are defined initially in the domain $D_k(u,l),\ D_k(u,l) \subset DB_k(u,l)$, of the billiards which boundary $\partial D_k(u,l)$ contains of course $A_k(u,l)$. The remaining part of $\partial D_k(u,l)$ is built of the two "limit" rays of $B_k(u,l)$ emerging from the ends of $A_k(u,l)$ and of $BA_k(u,l)$ if there is no caustic of the bundle $B_k(u,l)$ inside the billiards or by the corresponding caustic $K_k(u,l) = \{(f_k(K_k(s;u,l),s;u,l),\ g_k(K_k(s;u,l),s;u,l)) : J_k(K_k(s;u,l),s;u,l) = 0,\ \mathbf{r}_0(s) \in A_k(u,l)\}$.

Similarly $\Psi_{A;k}^\sigma(d,s;u,l;\lambda),\ \sigma = \pm$, is defined in the domain $D_k^A(u,l)$ corresponding to the bundle $B_k^A(u,l)$.

An important property of the representation (22) is its uniqueness, i.e. for two different bundles defined on the segment $A_k(u,l)$ this representation provides us with two different pairs of $\Psi_k^\sigma(d,s;u,l;\lambda),\ \sigma = \pm$. This conclusion follows from the fact that for λ sufficiently large the BSWF's are determined only by exponentials and the latter are different at the same points (x,y) for different bundles.

4.2 SWF's vanishing on the billiards boundary

Another obvious property of BSWF's $\Psi_k^\sigma(d,s;u,l;\lambda),\ \sigma = \pm$, is that they cannot vanish on $A_k(u,l)$ unless $\chi_k^\sigma(d,s;u,l;\lambda)$ vanish there identically. Therefore a wave function $\Psi_k^{as;\sigma}(x,y;u,l;\lambda)$ vanishing on $A_k(u,l)$ should be represented in the semiclassical limit by a linear combination of at least two BSWF's of the form (22). It is shown in App.A that the proper linear combinations have to be the following:

$$\Psi_k^{as;\sigma}(x,y;u,l;\lambda) = \Psi_k^\sigma(d_1,s_1;u,l;\lambda) + \Psi_{A;k}^{-\sigma}(d_2,s_2;u,l;\lambda) =$$
$$J_k^{-\frac{1}{2}}(d_1,s_1;u,l)e^{\sigma ik\left(d_1+\int_u^{s_1}\cos\alpha_k(s';u,l)ds'\right)}\chi_k^\sigma(d_1,s_1;u,l;\lambda) +$$
$$J_{A;k}^{-\frac{1}{2}}(d_2,s_2;u,l)e^{-\sigma ik\left(d_2-\int_u^{s_2}\cos\alpha_k(s';u,l)ds'\right)}\chi_{A;k}^{-\sigma}(d_2,s_2;u,l;\lambda) \qquad (28)$$

with the following boundary conditions:

$$\chi_k^\sigma(0,s;u,l;\lambda) + \chi_{A;k}^{-\sigma}(0,s;u,l;\lambda) = 0 \qquad \mathbf{r}_0(s) \in A_k(u,l) \qquad (29)$$

while the point (x, y) is the cross point of the respective trajectories belonging to different bundles, i.e.

$$\begin{aligned}\mathbf{r} \equiv [x, y] = \mathbf{r}_k(d_1, s_1; u, l) = \mathbf{r}_0(s_1) + \mathbf{d}_1(s_1; u, l) = \\ \mathbf{r}_{A;k}(d_2, s_2; u, l) = \mathbf{r}_0(s_2) + \mathbf{d}_2(s_2; u, l) \\ \mathbf{r}_k(d, s; u, l) \in B_k(u, l), \qquad \mathbf{r}_{A;k}(d, s; u, l) \in B_k^A(u, l)\end{aligned} \tag{30}$$

The vanishing superposition (28) if defined on the bundle $B_{k'}(u', l')$ suggests that $\Psi^{-\sigma}_{A;k'}(d, s; u', l'; \lambda)$ should be related somehow to the BSWF $\Psi^{\sigma}_k(d, h_k^{-1}(s; u, l); u, l); u, l; \lambda)$ defined on the bundle $B_k(u, l)$ which the previous one is a reflection. In the next section this relation is established as a condition matching both the solutions.

4.3 SWF's defined on a bundle skeleton and their continuity

According to our construction of the skeletons $\mathbf{B}$ and $\mathbf{B}^A$ there is a domain $D_{\partial B}$, $B \supset D_{\partial B} \supset \partial B$, of the billiards containing the billiards boundary ∂B in which each point (x, y) with its some small vicinities is mapped in the one-to-one way into each bundle of the pairs $B_k(u, l)$ and $B_k^A(u, l)$ containing this point.

Let $\mathbf{r} = (x, y) \in D_{\partial B}$ be a fixed point of the billiards. Let $D(x, y)$ denote a set of all $D_k(u, l)$, $B_k(u, l) \in \mathbf{B}$, which contain this point and $D_A(x, y)$ is the respective set of $D_k^A(u, l)$, $B_k^A(u, l) \in \mathbf{B}^A$. According to their definition $D_k(u, l) \in D(x, y)$ if and only if $D_k^A(u, l) \in D_A(x, y)$.

SWF's $\Psi^{as;\sigma}_{\mathbf{B}}(x, y, \lambda)$ vanishing on the billiards boundary can now be defined on $\mathbf{B}$ and $\mathbf{B}^A$ in the domain $D_{\partial B}$ as follows:

$$\begin{aligned}\Psi^{as;\sigma}_{\mathbf{B}}(x, y, \lambda) = \sum_{\substack{D_k(u,l) \in D(x,y) \\ D_k^A(u,l) \in D_A(x,y)}} (\Psi^{\sigma}_k(d(u, l), s(u, l); u, l; \lambda) + \\ \Psi^{-\sigma}_{A;k}(d(u, l), s(u, l); u, l; \lambda))\end{aligned} \tag{31}$$

where BSWF's $\Psi^{\sigma}_k(d(u, l), s(u, l); u, l; \lambda)$ and $\Psi^{-\sigma}_{A;k}(d(u, l), s(u, l); u, l; \lambda)$ are defined in $D_{\partial B}$ and satisfy the condition (29) on $A_k(u, l)$.

The solutions $\Psi^{as;+}_{\mathbf{B}}(x, y, \lambda)$ and $\Psi^{as;-}_{\mathbf{B}}(x, y, \lambda)$ coincide if and only if $\mathbf{B} = \mathbf{B}^A$.

The SWF's (31) which satisfy the condition of vanishing on the billiards boundary are the most general ones for the skeletons $\mathbf{B}$ and $\mathbf{B}^A$ which can be defined in the domain $D_{\partial B}$. However the next step in solving the basic problem of energy quantization in the semiclassical limit is to make these solutions continuous in $D_{\partial B}$ since this property is not ensured automatically by (31). $\Psi^{as;\sigma}_{\mathbf{B}}(x, y, \lambda)$ are certainly continuous and unique inside each bundle contained in $D_{\partial B}$. However if the skeleton $\mathbf{B}$ is singular then their bundles have their boundaries also inside the billiards area on which $\Psi^{as;\sigma}_{\mathbf{B}}(x, y, \lambda)$ or their derivatives can appear to be discontinuous if a point (x, y) crosses these boundaries. They can be also non unique if a point (x, y) moves along some closed loops such as the one which is homotopic with the billiards boundary, i.e. it can happen that $\Psi^{as;\sigma}_{\mathbf{B}}(x, y, \lambda)$ do not come back to their initial values being continued along such a loop.

Considering the continuity property of $\Psi^{as;\sigma}_{\mathbf{B}}(x, y, \lambda)$ the following circumstances can accompany in general such bundle boundary crossings:

1. two BSWF's $\Psi_k^{\pm}(d,s;u,l;\lambda)$ and $\Psi_{k'}^{\pm}(d',s';u',l';\lambda)$ which enter the sum (31) are defined on the bundles $B_k(u,l)$ and $B_{k'}(u',l')$ which can be composed into a compound bundle; and

2. there is no such BSWF's and the respective neighboring bundles.

In the first of the above cases the corresponding $\Psi_k^{\pm}(d,s;u,l;\lambda)$, $(x_k(d,s),y_k(d,s)) \in DB_k(u,l)$, and $\Psi_{k'}^{\pm}(d',s';u',l';\lambda)$, $(x_{k'}(d',s'),y_{k'}(d',s')) \in DB_{k'}(u',l')$, defined by (26) have to be identified on the common boundary of the bundles $B_k(u,l)$ and $B_{k'}(u',l')$ together with their first derivatives, i.e.

$$\begin{aligned}
\Psi_k^{\pm}(d,s;u,l;\lambda) &= \Psi_{k'}^{\pm}(d',s';u',l';\lambda)\\
\frac{\partial}{\partial x}\Psi_k^{\pm}(d,s;u,l;\lambda) &= \frac{\partial}{\partial x}\Psi_{k'}^{\pm}(d',s';u',l';\lambda)\\
\frac{\partial}{\partial y}\Psi_k^{\pm}(d,l;u,l;\lambda) &= \frac{\partial}{\partial y}\Psi_{k'}^{\pm}(d',s';u',l';\lambda)\\
(x_k(d,l),y_k(d,l)) \equiv (x_{k'}(d',s'),y_{k'}(d',s')) &\in \partial DB_k(u,l)\cap\partial DB_{k'}(u',l') \neq \oslash
\end{aligned} \tag{32}$$

Note that such identifications as the last ones mean that the BSWF's are defined now on the reduced skeleton $\mathbf{B}^R$ rather then on the original ones.

In the second case however the corresponding BSWF's $\Psi_k^{\pm}(d,s;u,l;\lambda)$, $(x(d,s),y(d,s)) \in DB_k(u,l)$ or their normal derivatives have to vanish on such a boundary of $DB_k(u,l)$, i.e.

$$\begin{aligned}
&\Psi_k^{\pm}(d,s;u,l;\lambda) = 0\\
&(x_k(d,s),y_k(d,s)) \in \partial DB_k(u,l)
\end{aligned} \tag{33}$$

or

$$\begin{aligned}
&\frac{\partial}{\partial n}\Psi_k^{\pm}(d,s;u,l;\lambda) = 0\\
&(x(d,s),y(d,s)) \in \partial DB_k(u,l)
\end{aligned} \tag{34}$$

that is in such cases $\Psi_k^{\pm}(d,s;u,l;\lambda)$ defined in the bundle $B_k(u,l)$ should satisfy on its boundary Dirichlet's or Neumann's conditions.

The last two conditions though necessary seem to look as a little bit arbitrary. However we should remember that our calculations are performed in the semiclassical regime, i.e. in the classically allowed regions (bundles) outside which the semiclassical wave functions cannot exist. Physically this means obviously that outside each bundle a corresponding piece of the exact wave function represented on the bundle by its respective semiclassical approximation has to vanish exponentially (for λ sufficiently large) when moving away from the bundle. Semiclassically it just means that BSWF's defined inside the bundles have to vanish identically outside of them. This condition can course however that the first derivatives of the global SWF's (31) can be discontinuous on such bundles boundaries. Just this last property differs essentially the semiclassical solutions (31) from the exact ones. If it happens we will call such a GSWF **singular** in contrast to the **regular** one which is continuous in the whole billiards together with its first derivatives. From this discussion it follows also that for the latter possibility to happen it is necessary for the skeleton on which $\Psi_{\mathbf{B}}^{as;\sigma}(x,y,\lambda)$ is defined to be global.

4.4 First quantization condition for SWF's

If $\Psi_{\mathbf{B}}^{as;\sigma}(x, y, \lambda)$ are made continuous inside $D_{\partial B}$ then the uniqueness condition for them on each loop lying in $D_{\partial B}$ particularly on the ones homotopic with the billiards boundary ∂B if such exists leads us to the first quantization condition which has to be satisfied by these two SWF's.

4.5 Continuing the SWF's $\Psi_{\mathbf{B}}^{as;\sigma}(x, y, \lambda)$ over the whole skeletons B and $\mathbf{B}^A$ - the global SWF's

By the formula (31) $\Psi_{\mathbf{B}}^{as;\sigma}(x, y, \lambda)$ are defined in every bundle of the skeletons $\mathbf{B}$ and $\mathbf{B}^A$ close to the billiards boundary. However by the way of construction of both the skeletons if a point (x, y) of the domain $D_{\partial B}$ is achieved by a ray of some bundle of $\mathbf{B}$ running all the time by the domain $D_{\partial B}$ then it is also achieved by the same ray inverted in time and being a member of a bundle of $\mathbf{B}^A$, i.e. running in the opposite direction. However the second ray to achieve the considered point $(x, y) \in D_{\partial B}$ has in general first to leave the domain $D_{\partial B}$ crossing its boundary in several points in order to come back to it.

The same note is valid for rays contained in the skeleton $\mathbf{B}^A$.

Therefore $\Psi_{\mathbf{B}}^{as;\sigma}(x, y, \lambda)$ defined by the formula (31) can be continued from a point (x, y) of the domain $D_{\partial B}$ into another such point of $D_{\partial B}$ along rays contained in the skeletons $\mathbf{B}$ or $\mathbf{B}^A$. If $D_{\partial B}$ cannot be equal B, then such a continuation meet as necessary caustic points which have to be avoided somehow. If we do that however we will achieve again points of the domain $D_{\partial B}$ and naturally the continued solutions and the solutions defined by (31) have to coincide. This coincidence formulate the second quantization condition which both the SWF's $\Psi_{\mathbf{B}}^{as;\sigma}(x, y, \lambda)$ have to satisfy. Such a coincidence is achieved by identifying each term of the sum (31) with the corresponding term of the continued $\Psi_{\mathbf{B}}^{as;\sigma}(x, y, \lambda)$. Anticipating the results of App.B of [22] the corresponding identification should be done as follows.

1. Let $B_k(u, l)$ be a reflection of the bundles $B_{k_1}(u_1, l_1), B_{k_2}(u_2, l_2), \ldots, B_{k_n}(u_n, l_n)$ satisfying (20). Let $\Psi_{k_1}^{\sigma;cont}(d, s; u_1, l_1; \lambda), \Psi_{k_2}^{\sigma;cont}(d, s; u_2, l_2; \lambda), \ldots, \Psi_{k_n}^{\sigma;cont}(d, s; u_n, l_n; \lambda)$ denote the SWF's defined in $D_{\partial B}$ and continued on the respective bundles $B_{k_1}(u_1, l_1)$, $B_{k_2}(u_2, l_2)$, $\ldots, B_{k_n}(u_n, l_n)$ again to $D_{\partial B}$. Let further $\Psi_{A;k}^{-\sigma}(d, s; u, l; \lambda)$ be defined in $D_{\partial B}$ on the bundle $B_k^A(u, l)$ while $\Psi_k^{\sigma}(d, s; u, l; \lambda)$ in $D_{\partial B}$ on the bundle $B_k(u, l)$ being both related by the boundary condition (29).

   Then we make the following identification of BSWF's:

$$\begin{aligned} \Psi_{A;k}^{-\sigma}(d, h(s; u_j, l_j); u, l; \lambda) = \Psi_{k_j}^{\sigma;cont}(D(s; u_j, l_j) - d, s; u_j, l_j; \lambda) \\ \mathbf{r}_0(h(s; u_j, l_j)) \in A_k(u, l) \cap BL(u_j, l_j) \\ \mathbf{r}_0(s) \in L(u_j, l_j) \\ \mathbf{r}_0(h(s; u_j, l_j)) = \mathbf{r}_0(s) + \mathbf{D}(s; u_j, l_j), \quad j = 1, ...n \end{aligned} \tag{35}$$

   where $D(s; u, l) = |\mathbf{D}(s; u, l)|$ denotes the distance between the points $\mathbf{r}_0(h_k(s; u, l))$ and $\mathbf{r}_0(s)$ of the boundary ∂B.

   Similarly

2. Let $B_k^A(u, l)$ be a reflection of the bundles $B_{j_1}^A(u'_1, l'_1), B_{j_2}^A(u'_2, l'_2), \ldots, B_{j_m}^A(u'_m, l'_m)$ satisfying (20). Let $\Psi_{A;j_1}^{\sigma;cont}(d, s; u'_1, l'_1; \lambda), \Psi_{A;j_2}^{\sigma;cont}(d, s; u'_2, l'_2; \lambda), \ldots, \Psi_{A;k_n}^{\sigma;cont}(d, s; u'_m, l'_m; \lambda)$

denote the SWF's defined in $D_{\partial B}$ and continued on the respective bundles $B^A_{j1}(u'_1,l'_1)$, $B^A_{j2}(u'_2,l'_2),\ldots,B^A_{jm}(u'_m,l'_m)$ again to $D_{\partial B}$. Let further $\Psi_k^{-\sigma}(d,s;u,l;\lambda)$ be defined in $D_{\partial B}$ on the bundle $B_k(u,l)$ while $\Psi^\sigma_{A;k}(d,s;u,l;\lambda)$ in $D_{\partial B}$ on the bundle $B^A_k(u,l)$ being both related by the boundary condition (29).

Then we make the following identification:

$$\begin{aligned}\Psi_k^{-\sigma}(d,h(s;u_i,l_i);u,l;\lambda) = \Psi^{\sigma;cont}_{A;j_i}(D(s;u_i,l_i)-d,s;u_i,l_i;\lambda)\\ \mathbf{r}_0(h(s;u_i,l_i))\in A_k(u,l)\cap BL(u_i,l_i)\\ \mathbf{r}_0(s)\in L(u_i,l_i)\\ \mathbf{r}_0(h(s;u_i,l_i)) = \mathbf{r}_0(s)+\mathbf{D}(s;u_i,l_i), \qquad i=1,...m\end{aligned} \tag{36}$$

3. Meeting the caustic points the BSWF's $\Psi^\sigma_k(d,s;u,l;\lambda)$ and $\Psi^\sigma_{A;k}(d,s;u,l;\lambda)$ avoid them by fixing s and moving on the complex d-plane from above the points for $\sigma=+$ and from below them for $\sigma=-$.

The conditions (35) - (36) allow us to define $\Psi^{as;\sigma}_{\mathbf{B}}(x,y,\lambda)$ as given by (31) in every point of the domain $D_{\mathbf{B}}$, i.e. in the domain classically allowed when moving on the skeleton $\mathbf{B}$ and to rewrite the sum in (31) representing $\Psi^{as;\sigma}_{\mathbf{B}}(x,y,\lambda)$ globally by the terms of $\Psi^\sigma_k(d(u,l),s(u,l);u,l;\lambda)$ or by the terms of $\Psi^{-\sigma}_{A;k}(d(u,l),s(u,l);u,l;\lambda)$. Namely, the global SWF's (GSWF) $\Psi^{as;\sigma}_{\mathbf{B}}(x,y,\lambda)$ are given by:

$$\begin{aligned}\Psi^{as;\sigma}_{\mathbf{B}}(x,y,\lambda) = \sum_{DB_k(u,l)\in\tilde{D}(x,y)}\Psi^\sigma_k(d(u,l),s(u,l);u,l;\lambda) =\\ \sum_{DB^A_k(u,l)\in\tilde{D}_A(x,y)}\Psi^{-\sigma}_{A;k}(d(u,l),s(u,l);u,l;\lambda)\end{aligned} \tag{37}$$

where $\tilde{D}(x,y)$ and $\tilde{D}_A(x,y)$ denote now the respective sets of $DB_k(u,l)$ and $DB^A_k(u,l)$ containing the point (x,y).

The sums in (37) contain all BSWF's $\Psi^\sigma_k(d,s;u,l;\lambda)$ and $\Psi^{-\sigma}_{A;k}(d,s;u,l;\lambda)$ which can be continued to this point by the corresponding domains $DB_k(u,l)$ and $DB^A_k(u,l)$.

Let us note that the forms (37) of the GSWF's allow us in fact to define them on the reduced form $\mathbf{B}^R$ of the skeleton $\mathbf{B}$ rather then on the skeleton itself. This possibility permits to reduce substantially number of terms in sums (37).

Rewritten in terms of the χ-coefficients Eq.(35) gives:

$$\begin{aligned}\chi^{-\sigma}_{A;k}(d,h(s;u_j,l_j);u,l;\lambda) =\\ \eta_\sigma e^{\sigma i\lambda p\delta_k(u_j,l_j)}\left|\frac{\partial h(s;u_j,l_j)}{\partial s}\right|^{-\frac{1}{2}}\chi^{\sigma;cont}_{k_j}(D(s;u_j,l_j)-d,s;u_j,l_j;\lambda)\\ \delta_k(u_j,l_j) = D(s;u_j,l_j)+\int_{u_j}^s\cos\alpha_{k_j}(s';u_j,l_j)ds' - \int_u^{h(s;u_j,l_j)}\cos\alpha_k(s';u,l)ds'\\ \mathbf{r}_0(h(s;u_j,l_j))\in A_k(u,l)\cap BL(u_j,l_j)\\ \mathbf{r}_0(s)\in L(u_j,l_j)\\ \mathbf{r}_0(h(s;u_j,l_j)) = \mathbf{r}_0(s)+\mathbf{D}(s;u_j,l_j), \qquad j=1,...n\end{aligned} \tag{38}$$

Note that $\delta_k(u_j,l_j)$ in the above formula is s-independent (see App.B). Due to that and due to the properties (16) and (17) the rhs of (38) satisfies (12) as it should.

Putting $d = 0$ in (38) and taking into account (29) we get:

$$\chi_k^\sigma(0,h(s;u_j,l_j);u,l;\lambda) = -\chi_{A;k}^{-\sigma}(0,h(s;u_j,l_j);u,l;\lambda) =$$
$$-\eta_\sigma e^{\sigma i\lambda p\delta_k(u_j,l_j)}\left|\frac{\partial h(s;u_j,l_j)}{\partial s}\right|^{-\frac{1}{2}}\chi_{k_j}^{\sigma;cont}(D(s;u_j,l_j),s;u_j,l_j;\lambda) =$$
$$-\eta_\sigma e^{\sigma i\lambda p\delta_k(u_j,l_j)}\chi_{k_j}^{\sigma,cont}(0,h(s;u_j,l_j);u,l;\lambda)$$
$$\mathbf{r}_0(h(s;u_j,l_j)) \in A_k(u,l)\cap BL(u_j,l_j)$$
$$\mathbf{r}_0(s) \in L(u_j,l_j)$$
$$\mathbf{r}_0(h(s;u_j,l_j)) = \mathbf{r}_0(s) + \mathbf{D}(s;u_j,l_j), \qquad j = 1,...n \tag{39}$$

The BSWF's $\Psi^\sigma(d,s;u,l;\lambda)$ defined on bundles of $\mathbf{B}$ and $\Psi_A^\sigma(d,s;u,l;\lambda)$ defined on respective bundles of $\mathbf{B}^A$ are related with each other by the boundary conditions (29) and by matching conditions (36)-(39).

It is clear that the conditions (39) have to determine also the χ-factors $\chi_k^\sigma(s;u,l;\lambda)$ for all the bundles $B_k(u,l)$ which are the "initial" conditions for both $\chi_k^\sigma(d,s;u,l;\lambda)$ and $\chi_{A;k}^\sigma(d,s;u,l;\lambda)$ in the recurrent formula (15), i.e. $\chi_k^\sigma(s;u,l;\lambda) \equiv \chi_k^\sigma(0,s;u,l;\lambda) \equiv -\chi_{A;k}^\sigma(0,s;u,l;\lambda)$. Nevertheless these conditions cannot be given arbitrarily. Just opposite all $\chi_k(s;u,l;\lambda)$ have to satisfy (39) in a selfconsistent way.

The formulae (38) and (39) define the conditions which the SWF's $\chi_k^\sigma(d,h(s;u_j,l_j);u,l;\lambda)$ should satisfy when bouncing from the billiards boundary. Nevertheless this condition can be specified additionally with respect to its factors. Namely, taking their large λ-limit we get:

$$\chi_{k,0}^\sigma(h(s;u_j,l_j);u,l) = -\eta_\sigma\left|\frac{\partial h(s;u_j,l_j)}{\partial s}\right|^{-\frac{1}{2}} e^{\sigma i\lambda p\delta_k(u_j,l_j)}\chi_{k_j,0}^\sigma(s;u_j,l_j)$$
$$\chi_{k,r+1}^\sigma(h(s;u_j,l_j);u,l) = -\eta_\sigma\left|\frac{\partial h(s;u_j,l_j)}{\partial s}\right|^{-\frac{1}{2}} e^{\sigma i\lambda p\delta_k(u_j,l_j)}\times$$
$$\left(\chi_{k_j,r+1}^\sigma(s;u_j,l_j) + \frac{\sigma i}{2p}\int_0^{D(s;u_j,l_j)}\left(\cdot J^{\frac{1}{2}}\triangle\left(J^{-\frac{1}{2}}\chi_{k_j,r}^\sigma(a,s;u_j,l_j)\right) + \right.\right.$$
$$\left.\left. 2\sum_{l=0}^r E_{r-l+1}\chi_{k_j,l}^\sigma(a,s;u_j,l_j)\right)da\right)$$
$$r = 0,1,2,..., \tag{40}$$

The above equations should be satisfied on each bundle $B_k(u,l)$ of the skeleton $\mathbf{B}$.

The first of the equations (40) should determine the classical quantities, namely the skeleton $\mathbf{B}$ and the "classical" energy $E_0 = \frac{1}{2}p^2$ and by them define the JWKB approximation of the SWF's. Namely:

$$\Psi_{\mathbf{B}}^{JWKB;\sigma}(x,y,\lambda) = \sum_{DB(u,l)\in D(x,y)}\Psi^{JWKB;\sigma}(d(u,l),s(u,l);u,l) =$$
$$\sum_{DB(u,l)\in D(x,y)} J^{-\frac{1}{2}}(d(u,l),s(u,l))e^{\sigma i\lambda p\left(d(u,l)+\int_u^{s(u,l)}\cos\alpha(s';u,l)ds'\right)}\chi_0^\sigma(d(u,l),s(u,l);u,l) \tag{41}$$

The remaining equations determine quantum corrections to the "classical" ones involved in (41).

However it is easy to note that for the selfconsistency of the equations (40) it is necessary for the exponent $e^{\sigma i\lambda p\delta_k(u_j,l_j)}$ to be independent of λ, i.e. we have to have on each bundle $B_k(u,l)$ of $\mathbf{B}$:

$$\begin{aligned} \lambda p\delta_k(u,l) &= \phi_k(u,l) \\ B_k(u,l) &\subset \mathbf{B} \end{aligned} \tag{42}$$

where $\delta_k(u,l)$ is given by (38) and $\phi_k(u,l)$ is a λ-independent constant.

The equations (42) have to define the energy $E_0 = \frac{1}{2}p^2$.

Taking into account the last conclusions we get the following final set of the recurrent quantization conditions:

$$\begin{aligned} \lambda p\delta_k(u,l) &= \phi_k(u,l) \\ \chi^\sigma_{k,0}(h(s;u_j,l_j);u,l) &= -\eta_\sigma e^{\sigma i\phi_{k_j}(u_j,l_j)}\left|\frac{\partial h(s;u_j,l_j)}{\partial s}\right|^{-\frac{1}{2}}\chi^\sigma_{k_j,0}(s;u_j,l_j) \\ \chi^\sigma_{k,r+1}(h(s;u_j,l_j);u,l) &= -\eta_\sigma e^{\sigma i\phi_{k_j}(u_j,l_j)}\left|\frac{\partial h(s;u_j,l_j)}{\partial s}\right|^{-\frac{1}{2}}\Bigg(\cdot\chi^\sigma_{k_j,r+1}(s;u_j,l_j)+ \\ \frac{\sigma i}{2p}\int_0^{D(s;u_j,l_j)} &\left(\cdot J^{\frac{1}{2}}\triangle\left(J^{-\frac{1}{2}}\chi^\sigma_{k_j,r}(a,s;u_j,l_j)\right)+2\sum_{l=0}^{r}E_{r-l+1}\chi^\sigma_{k_j,l}(a,s;u_j,l_j)\right)da\Bigg) \\ & r = 0,1,2,..., \end{aligned} \tag{43}$$

together with:

$$\begin{aligned} \chi^\sigma_{k,0}(d,s;u,l) &\equiv \chi^\sigma_{k,0}(s;u,l) \\ \chi^\sigma_{k,r+1}(d,s;u,l) &= \chi^\sigma_{k,r+1}(s;u,l) + \\ \frac{\sigma i}{2p}\int_0^d &\left(J^{\frac{1}{2}}\triangle\left(J^{-\frac{1}{2}}\chi^\sigma_{k,r}(a,s;u,l)\right)+2\sum_{m=0}^{r}E_{r-l+1}\chi^\sigma_{k,m}(a,s;u,l)\right)da \\ & r = 0,1,2,..., \end{aligned} \tag{44}$$

Let us note finally that if $\mathbf{B} \neq \mathbf{B}^A$ then energy levels corresponding to the skeleton $\mathbf{B}$ have to be degenerate. This conclusion follows easily from the form of the quantization conditions (43)-(44) and (32)-(33) showing that the complex conjugations of $\Psi^{as;\sigma}_{\mathbf{B}}(x,y,\lambda)$ satisfy also these conditions with the same semiclassical energy E. The two corresponding solutions are of course $\Psi^{as;\pm}_{\mathbf{B}}(x,y,\lambda)$.

4.6 Finite and infinite bundle structures of skeletons. The last quantization condition

The constructions of skeletons and SWF's in billiards performed in sec.3-4 describe completely the energy quantization problem in the semiclassical approximation.

For a given billiards however there can be skeletons with a finite number of bundles as well as with an infinite one. The semiclassical quantization procedure described in the previous sections seems to be easily applied to the finite bundle number skeletons. Namely in such a case following a trajectory starting from a bundle $B_k(u,l)$ we have to approach the same bundle after a finite number of bounces. The corresponding semiclassical wave function propagated by the skeleton has therefore to come back to its initial form achieving again the

initial bundle. This condition closes essentially the process of quantization formulated in the previous sections. The respective conditions are of course the following:

$$\exp\left(\sum_{B_k(u,l)\in\mathbf{B}}\delta_k(u,l)\right)\prod_{k=1}^{n}(-\eta_{\sigma_k})=1$$
$$\chi_k^{\sigma,cont}(D(s,s'),s;u,l;\lambda)=\chi_k^{\sigma}(s';u,l;\lambda),\qquad s,s'\in B_k(u,l)\tag{45}$$

where n is a number of bounces and $D(s,s')$ is the global distance passed by the billiards ball along the investigated trajectory.

Skeletons with a finite number of bundles are typical for the billiards with the integrable or pseudointegrable motions. Nevertheless they can be found also as particular cases of motions in chaotic billiards as well.

The cases of skeletons with an infinite number of bundles are clearly much more difficult for investigations. Such skeletons should be typical for chaotic billiards.

According to its definition a bundles $B_k(u,l)$ can bifurcate after the reflection by the billiards boundary into many different subbundles, i.e. parts of other bundles having their beginnings also partly on the arc $BA_k(u,l)$. In fact a general behaviour of a skeleton in such chaotic cases should not differ essentially by its chaotic complexity from a chaotic trajectory reminding however rather a gigantic road-knot with infinitely many viaducts spanning the billiards boundary on which the billiards ball moves. It is obvious that if they exist their identification seems to be not an easy task.

Nevertheless the rule (46) can appear to be useful also even in such cases. This is because a ray beginning with a bundle $B_k(u,l)$ can come back to it even arbitrarily close to its initial starting point on $A_k(u,l)$ (according to the Poincare theorem) not repeating its way. But this is enough for writing the "last quantization condition" (46) where the sum goes now over all bundles of the skeleton passed by the ray.

In the next two sections we shall focus on the finite number cases of bundles in skeletons, i.e. applying this procedure to the simplest well known cases of the polygon billiards not avoiding however billiards with chaotic motions such as the Bunimovich one.

5 The rational polygon billiards

A two dimensional rational polygon billiards are distinguished by their pseudointegrability [8]. As we have discussed it in sec.3 a phase space corresponding to a motion in such a billiards on a given skeleton consists of a finite number of pieces parallel to the billiards plane and orthogonal to the two momentum axes and corresponding each to the compound bundles the reduced skeleton. In the case when the corresponding skeleton is regular then by the gluing procedure described in sec.3.2 one can get [8, 23] a two dimensional compact closed surface with a genus g given by:

$$g=1+\frac{N}{4}\sum_{k=1}^{n}\frac{p_k-1}{q_k}\tag{46}$$

where N is the number of the compound bundles, n is the number of the polygon vertices, and $\pi\frac{p_k}{q_k}$ with integers $p_k,\ q_k$ relatively prime is the angle enclosed k-th vertex, $k=1,...,n$.

In other cases of the skeletons developed in the rational polygons one gets surfaces which do not provide us with closed surfaces in the phase space, i.e. such skeletons are singular. In particular such singular skeletons are developed by periodic trajectories.

Note that it is the polygon skeleton property that if it contains at least one periodic trajectory then all trajectories of such a skeleton are also periodic. In such a polygon periodic skeleton there are always two (and no more) periodic trajectories each of which starts from some vertex of the polygon and runs to another one. These two periodic trajectories have been called by Bogomolny and Schmit [17] as **singular diagonals** (SD's) while the skeleton itself as the periodic orbit channel (POC). Therefore each periodic skeleton is defined by two SD's.

A convenient way of representing motions in a polygon billiards can be obtain by unfolding the polygon by its repeating reflections in its sides on which the trajectory reflections are performed. A frequently complicated pattern of the real trajectories takes then a simple form of parallel straight lines on such unfolded polygons.

While a triangle is the simplest polygon its billiards properties are in general not as such. A motion in rational triangles can be integrable if $g = 1$ for them so that for the triangle angles $\pi\frac{p_i}{q_i}$, $i = 1, 2, 3$, as it follows from (46) we have to have:

$$\begin{aligned} p_i = 1, \qquad i = 1,2,3 \\ \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1 \end{aligned} \tag{47}$$

Several obvious solutions to (47) give the following triangle angles $\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}\right)$, $\left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$ for the integrable cases.

In fact it is rather a rectangular billiards and its variations which we call broken rectangle billiards which seems to demonstrate sometimes in a spectacular way most advantages of the skeleton approach developed in sec.3. Therefore we will firstly consider the cases just mentioned. The cases of the equilateral triangle and the pentagon billiards will be considered next.

5.1 The rectangular billiards

Consider therefore the rectangular billiards shown in Fig.2. This billiards is the canonical example of the energy quantization problem because of its easiness to be solved by the variable separation method. According to Fig.2 the well known solution to the problem is given by the following two equations:

$$\begin{aligned} \lambda p_x a = m\pi \\ \lambda p_y b = n\pi \\ m, n = 1, 2, 3, ... \end{aligned} \tag{48}$$

giving the energy:

$$E = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 = \frac{\pi^2}{2\lambda^2}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \tag{49}$$

and being the result of the following form of the (non-normalized) energy eigenfunctions:

$$\begin{aligned} \Psi(x,y) = 4\sin(\lambda p_x x)\sin(\lambda p_y y) = \\ e^{\lambda p_x x - \lambda p_y y} + e^{-\lambda p_x x + \lambda p_y y} - e^{\lambda p_x x + \lambda p_y y} - e^{-\lambda p_x x - \lambda p_y y} \end{aligned} \tag{50}$$

which have to vanish on the billiards boundary.

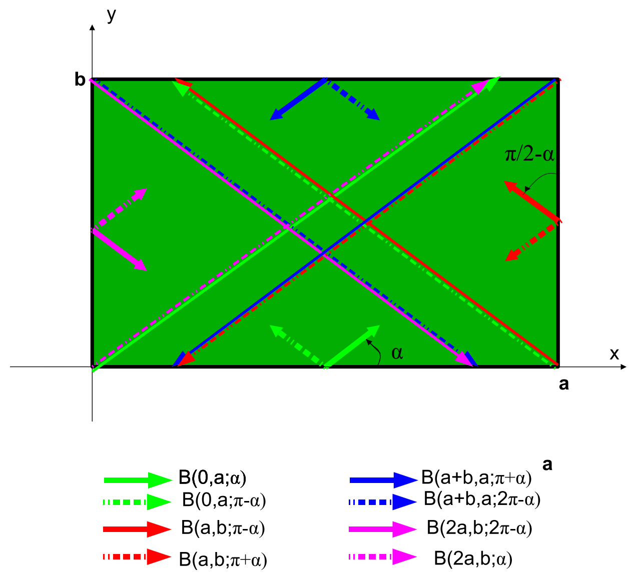


Figure 2: The eight ray bundles of the generic regular skeletons $\mathbf{B}$ and $\mathbf{B}^T(\equiv \mathbf{B})$ in the rectangular billiards and the one $(B_8 \cup B_1)$ of the four compound bundles. The latter form a torus in the phase space.

Of course one can always put $p_x = p\cos\alpha, p_y = p\sin\alpha$ where $\alpha,\ 0 < \alpha < \frac{1}{2}\pi$, is the angle by which the momentum p is inclined to the x-axis when the billiards ball reflects from the side A_1. Therefore the classical trajectory angles of the billiards ball are quantized according to the formula:

$$\begin{gathered}\tan\alpha = \frac{n}{m}\frac{a}{b}\\ m,n = 1,2,3,...\end{gathered} \tag{51}$$

Let us note that the cases $\alpha = 0, \frac{1}{2}\pi$ are excluded by the solutions (50).

Let us note further that the set Σ of all pairs $(m,n),\ m,n = 1,2,3,...,$ defining the eigenfunctions $\Psi_{m,n}(x,y)$ can be divided into disjoint subsets Σ_{m_0,n_0} each of which contains a pair (m_0,n_0) where m_0 and n_0 are relatively prime and all its multiples $(km_0,kn_0),\ k = 1,2,3,...,$ and nothing more so that $\Sigma = \bigcup_{m_0,n_0}\Sigma_{m_0,n_0}$. It is clear that all the points of Σ_{m_0,n_0} lie on the straight line $y = \tan\alpha_0 x$ with $\tan\alpha_0 = \frac{n_0}{m_0}\frac{a}{b}$, i.e. all the states $\Psi_{km_0,kn_0}(x,y),\ k = 1,2,3,...,$ are related in the rectangular billiards with a family of classical trajectories which are inclined to the x-axis by the angle α_0.

5.2 The rectangular billiards skeletons built by non-periodic trajectories

To perform semiclassical calculations corresponding to "generic" skeletons let us consider a skeleton shown in Fig.2 containing, by assumption, only nonperiodic trajectories. According to the description of the previous section there are four "smooth arcs" in the rectangular billiards, i.e. the four sides of the rectangle $A_1, ..., A_4$. Since the absolute values of the

momentum components p_x, p_y are the integrals of the classical motion inside the billiards respecting elastic law of bouncing then all bundles which should be taken into account are defined by a single angle α, $0 < \alpha < \frac{1}{2}\pi$, which are made by the rays of the bundle $B_1 = B_1(0, a; \alpha)$ with the x-axis.

Choosing the case of the angle α shown in Fig.2 the remaining seven bundles of the skeleton $\mathbf{B}$ shown in this figure are: $B_2 = B_1(0, a; \pi - \alpha)$, $B_3 = B_2(a, b; \pi - \alpha)$, $B_4 = B_2(a, b; \pi + \alpha)$, $B_5 = B_3(a + b, a; \pi + \alpha)$, $B_6 = B_3(a + b, a; 2\pi - \alpha)$, $B_7 = B_4(2a + b, b; 2\pi - \alpha)$, $B_8 = B_4(2a + b, b; \alpha)$, i.e. the parameter s introduced in sec.3 is counted anticlockwise starting from the point $(0, 0)$ of Fig.2 (and having negative value if measured clockwise). The bundles B_{2k-1}, B_{2k}, $k = 1, ..., 4$, are defined on the respective sides A_k, $k = 1, ..., 4$, of the billiards, i.e. on $A_1 = A_1(0, a)$, $A_2 = A_2(a, b)$, $A_3 = A_3(a + b, a)$, $A_4 = A_4(2a + b, b)$.

The skeleton $\mathbf{B}^T$ coincides exactly with $\mathbf{B}$ in the case of generic skeletons in the rectangular billiards, i.e. the corresponding energy levels cannot be degenerate.

Let us note that a number of bundles in the skeletons is obviously independent of a choice of α, i.e. it is always equal to eight if $0 < \alpha < \frac{1}{2}\pi$.

The corresponding Jacobean factors of $\Psi_q^{\pm}(d, s, \lambda)$, $q = 1, ..., 8$, are $J^{-\frac{1}{2}}(d, s) \equiv (-\sin\alpha_k)^{-\frac{1}{2}}$, $\alpha_k = \alpha$, $s \in A_k$, $k = 1, 3$ and $\alpha_k = \frac{1}{2}\pi - \alpha$, $s \in A_k$, $k = 2, 4$, i.e. the Jacobeans are constant but discontinues. Therefore they will be included into the χ-factors contained in the SWF's.

We can now make use of the fact that from the sixteen BSWF's $\Psi_q^{\pm}(d, s, \lambda)$, $q = 1, ..., 8$, we can first select only eight of them with the positive signature since the negative signature solutions have to coincide with the respective positive signature ones. Next since each pair $\Psi_{2q}^{+}(d, s, \lambda)$, $\Psi_{2q+1}^{+}(d, s, \lambda)$ $q = 1, ..., 4$, of these solutions has to coincide on the common boundary of the respective bundles B_{2q} and B_{2q+1}, $q = 1, ..., 4$, then we can define the solutions on the respective compound bundles to get in this way only four BSWF's, namely:

$$
\begin{aligned}
\tilde{\Psi}_1^{+}(d, s, \lambda) &\equiv \begin{cases} \Psi_8^{+}(d, s, \lambda) & -b < s < 0 \\ \Psi_1^{+}(d, s, \lambda) & 0 < s < a \end{cases} \\
\tilde{\Psi}_2^{+}(d, s, \lambda) &\equiv \begin{cases} \Psi_2^{+}(d, s, \lambda) & 0 < s < a \\ \Psi_3^{+}(d, s, \lambda) & a < s < a + b \end{cases} \\
\tilde{\Psi}_3^{+}(d, s, \lambda) &\equiv \begin{cases} \Psi_4^{+}(d, s, \lambda) & a < s < a + b \\ \Psi_5^{+}(d, s, \lambda) & -a - b < s < -b \end{cases} \\
\tilde{\Psi}_4^{+}(d, s, \lambda) &\equiv \begin{cases} \Psi_6^{+}(d, s, \lambda) & -a - b < s < -b \\ \Psi_7^{+}(d, s, \lambda) & -b < s < 0 \end{cases}
\end{aligned} \tag{52}
$$

and also the respective four compound bundles:

$$
\begin{aligned}
\tilde{B}_1 &\equiv B_{8,1} = B_8 \cup B_1 \\
\tilde{B}_2 &\equiv B_{2,3} = B_2 \cup B_3 \\
\tilde{B}_3 &\equiv B_{4,5} = B_4 \cup B_5 \\
\tilde{B}_4 &\equiv B_{6,7} = B_6 \cup B_7
\end{aligned} \tag{53}
$$

on which the four solutions (52) are defined.

Therefore the reduced skeleton $\mathbf{B}^R$ contains four compound bundles (53). Note that each compound bundle $\tilde{B}_k$, $k = 1, ..., 4$, is regular so that the "generic" rectangular billiards skeleton $\mathbf{B}$ is also regular.

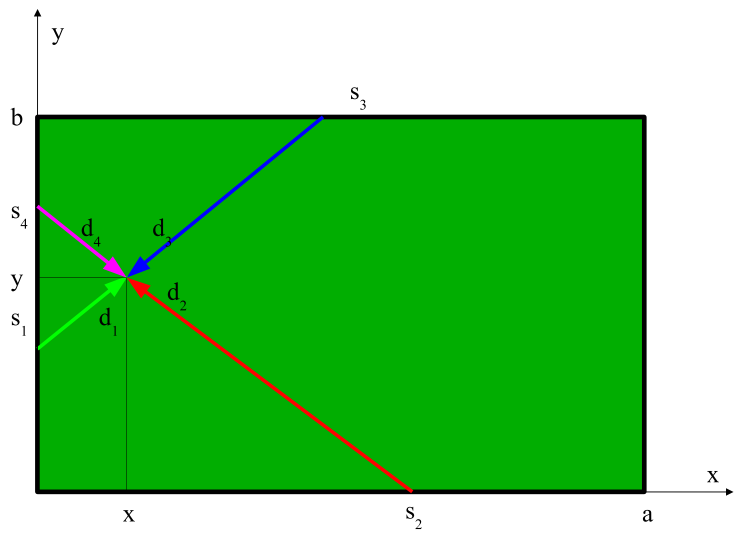


Figure 3: The four rays of the four compound bundles and the corresponding four solutions $\Psi_i^+(d_i, s_i),\ i = 1, ..., 4$, meeting at the point (x, y)

The above compound bundles are shown on Fig.3 where in each billiard point four their rays are met and the four solutions (52) are superposed to get GSWF, i.e.

$$\Psi^{as}(x, y) = \sum_{k=1}^{4} \tilde{\Psi}_k^+(d_k, s_k, \lambda) \tag{54}$$

Since in our further considerations we will work exceptionally with the solutions (52) we will drop the signature of these solutions as well as the tilde mark for a convenience. Then assume the following standard forms for $\Psi_k(d, s, \lambda),\ k = 1, ..., 4,$:

$$\Psi_k(d, s, \lambda) = e^{i\lambda pd + i\lambda ps \cos\alpha_k} \chi_k(d, s, \lambda) \tag{55}$$

where α_k is the angle the momentum of the ray of the compound bundle B_k makes with the corresponding side of the rectangle measured anticlockwise.

The solution $\Psi_k(d, s, \lambda)$ is defined on the compound bundle B_k which rays start from the sides A_{k-1} and A_k so that the variable s is measured from the left end of the corresponding side $A_k,\ k = 1, ..., 4$. For a given $\Psi_k(d, s, \lambda)$ s is then positive on A_k and negative on A_{k-1} where $\Psi_k(d, s, \lambda)$ is also defined. Since the constant Jacobean factors have been included into χ-coefficients the coefficient $\chi_k(d, s, \lambda)$ is continuous on the sides $A_{k-1} \cup A_k,\ k = 1, ..., 4$.

Let the colours of rays corresponding to the particular compound bundles B_k denote also colours of these bundles. Then unfolding the skeleton of Fig.3 onto the plane a motion of the billiards ball which begins with the rays of the bundle B_1 is limited by the stripe bounded by the two parallel (thick black) lines shown in Fig.4 and are performed along the straight line. It is seen clearly on the figure that this motion is just the scattering of the skeleton

bundles on the (white) vertices of the rectangle so that each bundle is scattered into the two neighbour ones with the exception of its single ray which crosses the vertex. The latter ray is scattered back into the third remaining bundle. No one of the straight line rays crosses the rectangle boundary at the same point and the crossing points of each ray are dense on each rectangle side. The ray of Fig.4 which starts at the point with the coordinate s on the figure is also shown in folded way on Fig.5.

The GSWF (54) has to vanish on each side of the rectangle and this condition exhausts all the conditions it has to satisfy. However, a particular form of the corresponding conditions depends on a choice of points on the rectangle billiards boundary even for the same bundle. Therefore let us choose for writing these conditions the four first points of the ray (including the starting point) shown in Fig.5. as convenient for our further considerations. As it follows from the form (54) of the solution and from Fig.4 and Fig.5 the corresponding conditions are:

$$\begin{aligned} e^{-i\lambda ps\sin\alpha}\chi_1(0,s,\lambda)+e^{i\lambda p(a\cos\alpha-s\sin\alpha)}\chi_2\left(\frac{-s}{\sin\alpha},-a-s\cot\alpha,\lambda\right)+ \\ e^{i\lambda p(a\cos\alpha+(b+s)\sin\alpha)}\chi_3\left(\frac{b+s}{\sin\alpha},a-(b+s)\cot\alpha,\lambda\right)+ \\ e^{i\lambda p(b+s)\sin\alpha}\chi_4(0,b+s,\lambda)=0 \end{aligned} \tag{56}$$

$$\begin{aligned} e^{i\lambda p(\frac{b}{\sin\alpha}+s\cot\alpha\cos\alpha)}\chi_1\left(\frac{b+s}{\sin\alpha},s,\lambda\right)+ \\ e^{i\lambda p(b\sin\alpha+(a-(b+s)\cot\alpha)\cos\alpha)}\chi_2\left(\frac{b}{\sin\alpha},(2b+s)\cot\alpha-a,\lambda\right)+ \\ e^{i\lambda p((a-(b+s)\cot\alpha)\cos\alpha)}\chi_3(0,a-(b+s)\cot\alpha,\lambda)+ \\ e^{i\lambda p(b+s)\cot\alpha\cos\alpha}\chi_4(0,-(b+s)\cot\alpha,\lambda)=0 \end{aligned} \tag{57}$$

$$\begin{aligned} e^{i\lambda p(2b+s)\cot\alpha\cos\alpha}\chi_1(0,(2b+s)\cot\alpha,\lambda)+ \\ e^{i\lambda p(a\cos\alpha-(2b+s)\cot\alpha\cos\alpha)}\chi_2(0,(2b+s)\cot\alpha-a,\lambda)+ \\ e^{i\lambda p(a\cos\alpha+b\sin\alpha-(2b+s)\cot\alpha\cos\alpha)}\chi_3\left(\frac{a-(2b+s)\cot\alpha}{\cos\alpha},3b+s-a\tan\alpha,\lambda\right)+ \\ e^{i\lambda p(\frac{b}{\sin\alpha}+(b+s)\cot\alpha\cos\alpha)}\chi_4\left(\frac{b}{\sin\alpha},-(b+s)\cot\alpha,\lambda\right)=0 \end{aligned} \tag{58}$$

$$\begin{aligned} e^{i\lambda p(\frac{a}{\cos\alpha}-(2b+s)\sin\alpha)}\chi_1\left(\frac{a-(2b+s)\cot\alpha}{\cos\alpha},(2b+s)\cot\alpha,\lambda\right)+ \\ e^{i\lambda p(a\tan\alpha\sin\alpha-(2b+s)\sin\alpha)}\chi_2(0,a\tan\alpha-2b-s,\lambda)+ \\ e^{i\lambda p(-a\tan\alpha\sin\alpha+(3b+s)\sin\alpha)}\chi_3(0,3b+s-a\tan\alpha,\lambda)+ \\ e^{i\lambda p((3b+s)\sin\alpha-a\tan\alpha\sin\alpha+a\cos\alpha)}\chi_4\left(\frac{3b+s-a\tan\alpha}{\sin\alpha},(3b+s)\cot\alpha-2a,\lambda\right)=0 \end{aligned} \tag{59}$$

where according to our convention s is negative being measured from the left end of the side A_1 of the rectangle.

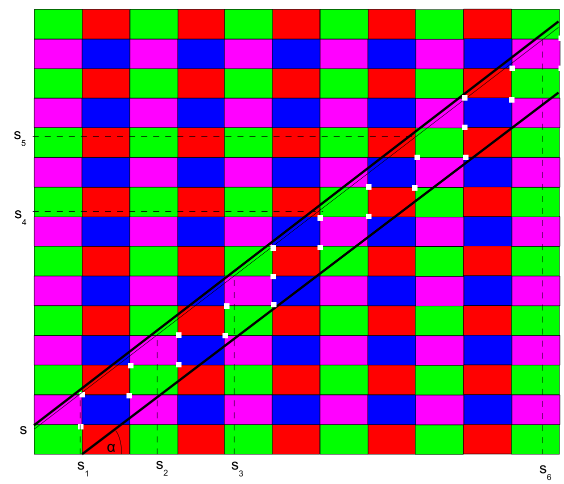


Figure 4: The unfolded motion in the rectangular billiards

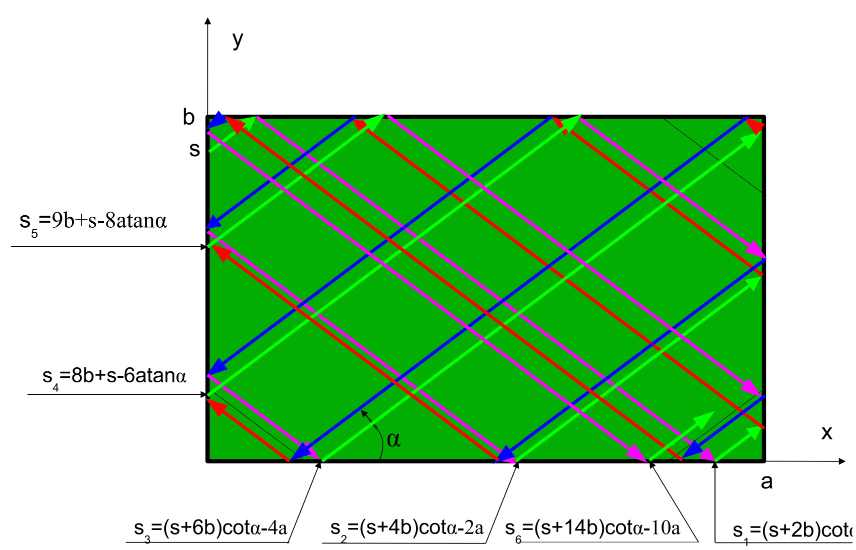


Figure 5: The solution $\Psi_1(d,s)$ being carried by the successive bundles $B_1 \to B_4 \to B_1 \to B_2 \to B_3 \to B_4 \to B_1 \to \ldots$. The corresponding ray is shown also unfolded in Fig.4

The last equations reduce to the following ones:

$$\chi_1(0,s,\lambda)+e^{i\lambda pa\cos\alpha}\chi_2\left(\frac{-s}{\sin\alpha},-a-s\cot\alpha,\lambda\right)=0$$
$$e^{i\lambda pa\cos\alpha}\chi_3\left(\frac{b+s}{\sin\alpha},a-(b+s)\cot\alpha,\lambda\right)+\chi_4(0,b+s,\lambda)=0 \tag{60}$$

$$e^{i\lambda pb\sin\alpha}\chi_1\left(\frac{b+s}{\sin\alpha},s,\lambda\right)+\chi_4(0,-(b+s)\cot\alpha,\lambda)=0$$
$$e^{i\lambda pb\sin\alpha}\chi_2\left(\frac{b}{\sin\alpha},(2b+s)\cot\alpha-a,\lambda\right)+\chi_3(0,a-(b+s)\cot\alpha,\lambda)=0 \tag{61}$$

$$\chi_1(0,(2b+s)\cot\alpha,\lambda)+e^{i\lambda pb\sin\alpha}\chi_4\left(\frac{b}{\sin\alpha},-(b+s)\cot\alpha,\lambda\right)=0$$
$$\chi_2(0,(2b+s)\cot\alpha-a,\lambda)+e^{i\lambda pb\sin\alpha}\chi_3\left(\frac{a-(2b+s)\cot\alpha}{\cos\alpha},3b+s-a\tan\alpha,\lambda\right)=0 \tag{62}$$

$$e^{i\lambda pa\cos\alpha}\chi_1\left(\frac{a-(2b+s)\cot\alpha}{\cos\alpha},(2b+s)\cot\alpha,\lambda\right)+\chi_2(0,a\tan\alpha-2b-s,\lambda)=0$$
$$\chi_3(0,3b+s-a\tan\alpha,\lambda)+e^{i\lambda pa\cos\alpha}\chi_4\left(\frac{3b+s-a\tan\alpha}{\sin\alpha},(3b+s)\cot\alpha-2a,\lambda\right)=0 \tag{63}$$

The first two equations of (61) and (62) can be "solved" in the first order in λ with the help of (15) to get:

$$\chi_{1,0}((2b+s)\cot\alpha)=e^{2i\lambda pb\sin\alpha}\chi_{1,0}(s) \tag{64}$$

The point $s=(2b+s)\cot\alpha$ lies now on the side A_1 of the rectangle. It is achieved by the ray leaving the starting point s of the side A_4 of the rectangle and passing the distance $\frac{2b+s}{\sin\alpha}$. Since the coefficient $\chi_{1,0}(s)$ does not change propagating along the ray we have to have:

$$\chi_{1,0}((2b+s)\cot\alpha)=\chi_{1,0}(s) \tag{65}$$

so that

$$e^{2i\lambda pb\sin\alpha}=1 \tag{66}$$

and

$$\lambda pb\sin\alpha=n\pi,\qquad n=1,2,.... \tag{67}$$

Similarly, choosing another propagation path we get:

$$e^{2i\lambda pa\cos\alpha}=1 \tag{68}$$

and

$$\lambda pa\cos\alpha = m\pi, \qquad m = 1, 2, ... \tag{69}$$

On the other hand the equation (65) is of great importance because it shows that $\chi_{1,0}(s)$ is defined in different points of the sides $A_1,\ A_4$ by its value established in some definite point of these sides and by the propagation procedure defined by (14) and (15). Since however a propagated ray reflects consecutively on the boundary in points densely distributed on it this initial value is also propagated densely on the boundary. Therefore to get the coefficient as a continuous function of s we have to put it a constant. Let it be equal to one.

The next order term propagates according to the formula:

$$\chi_{1,1}(d, s) = \chi_{1,1}(s) + \frac{iE_1}{p}d \tag{70}$$

where $\chi_{1,1}(s)$ is the initial value of the term on the boundary. Therefore, since distances d_i measured along the propagating ray of the consecutive boundary points s_i by which the ray is reflected are distributed on the boundary irregularly but densely values of $\chi_{1,1}(0, s_i) \equiv \chi_{1,1}(s_i) = \chi_{1,1}(d_i, s)$ in these points have to change discontinuously. Therefore to maintain the continuity property of $\chi_{1,1}(s)$ on the boundary we have to put $E_1 = 0$ in (70) so that $\chi_{1,1}(d, s)$ is again constant on the boundary and in consequence independent also of d.

Quite similarly we can argue that also the remaining terms have to be constant as well as all the other terms of the energy semiclassical series have to vanish.

In this way we get finally from (61)-(64):

$$\begin{aligned} \chi_{1,mn}(d, s, \lambda) &\equiv 1 \\ \chi_{2,mn}(d, s, \lambda) &\equiv (-1)^{m+1} \\ \chi_{3,mn}(d, s, \lambda) &\equiv (-1)^{m+n} \\ \chi_4(d, s, \lambda) &\equiv (-1)^{n+1} \end{aligned} \tag{71}$$

and

$$E_{mn} = E_{0,mn} = \frac{1}{2}p_{mn}^2 = \frac{\pi^2}{2\lambda^2}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right), \qquad m, n = 1, 2, 3, ... \tag{72}$$

Choosing therefore the point (x, y) of Fig.3 for the SWF (54) we get:

$$\Psi_{mn}^{as}(x, y) = \sum_{l=1}^{4}\Psi_l(d_l, s_l) = \sum_{l=1}^{4} e^{i\lambda p_{mn}d_l + i\lambda p_{mn}f_l(s_l) + \phi_l} \tag{73}$$

where d_l and $f_l(s_l),\ l = 1, ..., 4$ should be calculated from the relations (see Fig.5):

$$\begin{aligned} x &= x_0(s_l) + d_l\cos\alpha(s_l) \\ y &= y_0(s_l) + d_l\sin\alpha(s_l) \\ \alpha(s_l) &= \alpha,\ \frac{1}{2}\pi - \alpha \end{aligned} \tag{74}$$

and the phases ϕ_l are defined by (71).

However making use of the independence of the phase integral $\int_{(0,0)}^{(x,y)} p_x dx + p_y dy$ of the integration paths we get for the particular terms in the sum in (73):

$$\begin{aligned} e^{i\lambda pd_1+i\lambda pf_1(s_1)} &= e^{i\lambda(p_x x+p_y y)} \\ e^{i\lambda pd_2+i\lambda pf_2(s_2)+i(n+1)\pi} = e^{i\lambda(p_x(a-x)+p_y y+(n+1)\pi)} &= -e^{i\lambda(-p_x x+p_y y)} \\ e^{i\lambda pd_3+i\lambda pf_3(s_3)+i(m+n)\pi} = e^{i\lambda(p_x(a-x)+p_y(b-y)+(m+n)\pi)} &= e^{i\lambda(-p_x x-p_y y)} \\ e^{i\lambda pd_4+i\lambda pf_4(s_4)+i(m+1)} = e^{i\lambda(p_x x+p_y(b-y)+(m+1)\pi)} &= -e^{i\lambda(p_x x-p_y y)} \end{aligned} \quad (75)$$

where $p_x = p_{mn}\cos\alpha_{mn},\ p{=}p_{mn}\sin\alpha_{mn}$.

Finally:

$$\Psi_{mn}^{as}(x,y) = e^{i\lambda(p_x x+p_y y)} - e^{i\lambda(-p_x x+p_y y)} + e^{i\lambda(-p_x x-p_y y)} - e^{i\lambda(p_x x-p_y y)} = \\ -4\sin(p_x x)\sin(p_y y), \qquad m,n = 1,2,3,... \quad (76)$$

reproducing in this way the exact result (50).

Let us note that the set of all SWF's (76) is again complete.

Of course it is not surprising that the semiclassical calculations performed above reproduce the exact result (50) since it follows from its form that it represents simultaneously its semiclassical expansion. However to get the result (76) we have had to assume that skeletons considered had to be "generic", i.e. they were constructed of non-periodic trajectories. Just this assumption allows us to use the arguments of dens distributions of values of the SWF to establish its value on the boundary. In fact no other argument exists to get such a conclusion. On the other hand such an argument cannot be invoked in the cases of skeletons which are built of periodic trajectories so that such cases of skeletons must be considered separately.

Finally let us conclude that:

1. each non-periodic skeleton in the rectangular billiards is regular and equivalent in the phase space to a two dimensional torus;

2. the GSWF (76) is obviously regular and provides us with the exact solution to the SE;

3. the GSWF (76) as well as the corresponding energy levels coincide with their JWKB approximations since the corresponding semiclassical series for the SWF's and energy levels abbreviate on the zeroth term; and

4. it is the dense distribution of the skeleton rays in the configuration and the phase spaces which causes the semiclassical series abbreviation mentioned.

5.3 Skeletons built by periodic trajectories

Periodic skeletons in the rectangular billiards can be easily realized since bundles of such trajectories with a given period are defined by one of its members which starting from a vertex of the rectangle has to be reflected in other vertices to "finish" its motion in the initial vertex. Such a leading particular periodic trajectory will be called a singular diagonal (SD) after Bogomolny and Schmit [17]. All other trajectories which make the same angles with the corresponding sides of the rectangle as this particular SD does are then also periodic.

In the rational billiards periodic skeletons are defined always by two such SD's. In the rectangular billiard these two SD's are symmetric with respect to each other in a sense that

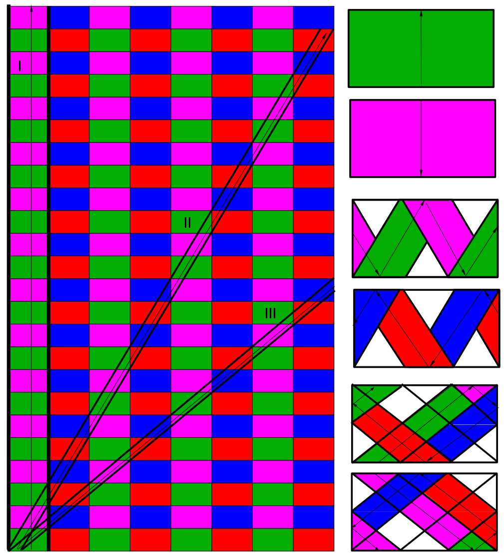


Figure 6: Three unfolded skeletons defined by the corresponding pairs of SD's. The skeletons II and III are singular. Every skeleton is the cylinder-like Lagrange surface in the phase space.

if one of them crosses some two different vertices of the rectangle the second one has to cross the remaining two.

On Fig.6 and Fig.7 there are shown five cases of skeletons defined by pairs of such SD's represented in their unfolded form (the left picture) and in their real form in the billiards (the right pictures). Each pair of SD's defining each skeleton are visible in the unfolded form of the skeletons as two parallel straight lines. Each skeleton is a stripe bounded by such two SD's. A general property of each such a stripe is that all the rectangle vertices related with the stripe lie on its boundary, i.e. on its two SD's.

Single periodic trajectories are shown also in each skeleton case in the figure being parallel to the SD's defining skeletons. In the billiards (the right pictures) these periodic trajectories are of course closed. The skeletons on the figure have forms which are typical, i.e. infinitely many others differ from these on the figure by a number of reflections of SD's on the rectangle sizes.

5.3.1 Bouncing ball skeleton and the corresponding regular GSWF's

We will construct GSWF's on these skeletons with the same rules as formulated earlier. Let us begin with the case of the skeleton numbered by I in Fig.6 and shown in Fig.8. This bouncing ball skeleton contains only two bundles B_1 and B_3 - the first one with its rays directed up and starting from the side A_1 and the second B_3 with rays directed down starting from the side A_3. The skeletons $\mathbf{B}^A$ and $\mathbf{B}^R$ are identical with $\mathbf{B}$ which is of course regular.

For these particular cases of bundles rays for both the bundles will be positioned by the same parameter s measuring a distance of a ray from the y-axis along the corresponding sides

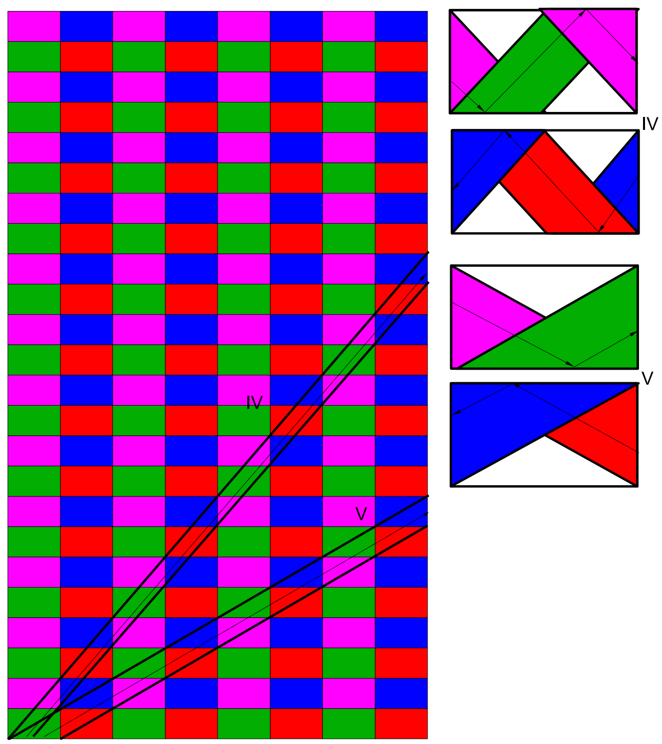


Figure 7: Another two unfolded singular skeletons defined by the corresponding pairs of SD's with the same cylinder-like Lagrange surfaces in the phase space.

A_1 and A_3. Therefore for the corresponding BSWF's:

$$\begin{aligned} \Psi_1(d,s,\lambda) &= e^{\lambda pd}\chi_1(d,s,\lambda) \\ \Psi_3(b-d,s,\lambda) &= e^{i\lambda p(b-d)}\chi_3(b-d,s,\lambda) \\ 0 \le d \le b, &\quad 0 < s < a \end{aligned} \tag{77}$$

For the coefficients $\chi_k(d,s,\lambda),\ k=1,3$ it is assumed as usually that they propagate along the rays of the bundles continuously and this their property is not influenced by reflections of the rays on the boundaries. Therefore we have to accept also that they are periodic with respect to the d-variable with the period equal to $2b$.

For the GSWF $\Psi^{as}(x,y,\lambda)$ we have therefore:

$$\Psi^{as}(x,y,\lambda) = \Psi_1(y,x,\lambda) + \Psi_3(b-y,x,\lambda) \tag{78}$$

together with the following boundary conditions on the sides A_1 and A_3 respectively:

$$\begin{aligned} \chi_1(0,x,\lambda) + e^{i\lambda pb}\chi_3(b,x,\lambda) = 0 \\ e^{i\lambda pb}\chi_1(b,x,\lambda) + \chi_3(0,x,\lambda) = 0 \end{aligned} \tag{79}$$

so that:

$$\chi_{1,0}(x) = e^{2i\lambda pb}\chi_{1,0}(x) \tag{80}$$

As previously we conclude that:

$$\lambda pb = n\pi, \qquad n = 1,2,3,... \tag{81}$$

Now the corresponding boundary conditions for $\Psi_k(y,x,\lambda),\ k=1,3$ on the sides A_2 and A_4 give:

$$\begin{aligned} \chi_k(y,0,\lambda) = \chi_k(y,a,\lambda) \equiv 0 \\ k = 1,3 \end{aligned} \tag{82}$$

Therefore in the zeroth order we have:

$$\chi_{1,0}(0) = \chi_{1,0}(a) = 0 \tag{83}$$

Next let us invoke the second of the equations (15) and the periodicity of $\chi_1(y,x,\lambda)$ to get in the considered case for the second order term:

$$\chi_{1,1}(2b,x) = \chi_{1,1}(0,x) = \chi_{1,1}(0,x) + \frac{ib}{2p}\left(\frac{d^2\chi_{1,0}(x)}{dx^2} + 2E_1\chi_{1,0}(x)\right) \tag{84}$$

so that

$$\frac{d^2\chi_{1,0}(x)}{dx^2} + 2E_1\chi_{1,0}(x) = 0 \tag{85}$$

The obvious solution of the last equation satisfying the boundary conditions (82) is:

$$\begin{aligned} \chi_{1,0}(x) = A_0\sin(\sqrt{2E_1}x) \\ \sqrt{2E_1}a = m\pi, \qquad m = 1,2,... \end{aligned} \tag{86}$$

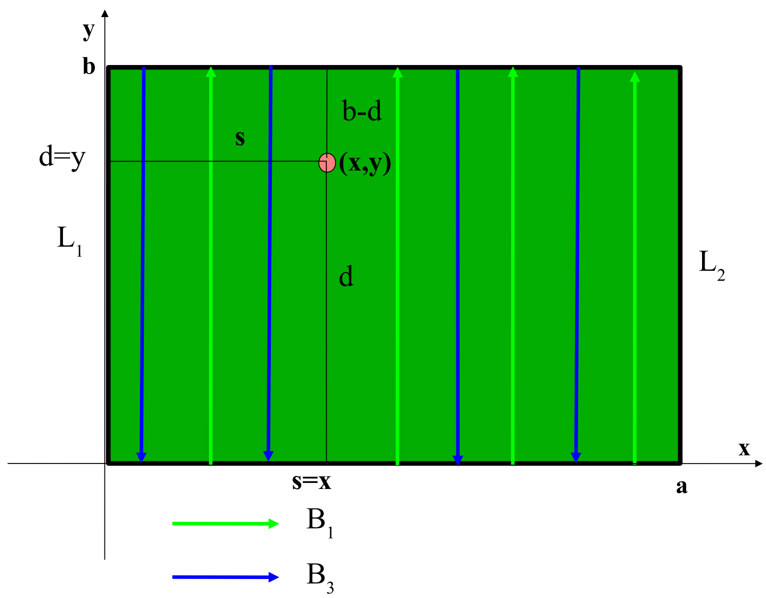


Figure 8: The two bouncing mode bundles B_1 and B_3 of the regular vertical skeleton in the rectangular billiards

Coming back to the second of the equations (15) we can conclude that $\chi_{1,1}(y,x)$ again is independent of y.

Passing next to the third of the equations (15) and repeating arguments similar to those which led us to (84) we get the following equation for $\chi_{1,1}(x)$:

$$\frac{d^2\chi_{1,1}(x)}{dx^2} + 2E_1\chi_{1,1}(x) + 2E_2\chi_{1,0}(x) = 0 \tag{87}$$

with the solution:

$$\chi_{1,1}(x) = A_1 \sin(\sqrt{2E_1}x) + B_1 \cos(\sqrt{2E_1}x) + \frac{E_2 A_0 x}{\sqrt{2E_1}} \cos(\sqrt{2E_1}x) \tag{88}$$

The boundary conditions $\chi_{1,1}(0) = \chi_{1,1}(a) = 0$ enforce however $B_1 = E_2 = 0$.

Using again (15) and the inductive arguments we come to the conclusion that $\chi_1(y,x,\lambda)$ is y-independent and the coefficients of its semiclassical series have the form:

$$\chi_{1,k}(x) = A_k \sin(\sqrt{2E_1}x), \qquad k = 0,1,... \tag{89}$$

so is the form of $\chi_1(x,\lambda)$ itself, i.e.

$$\begin{aligned} \chi_1(x,\lambda) = A(\lambda)\sin(\sqrt{2E_1}x) = A(\lambda)\sin\left(m\pi\frac{x}{a}\right) \\ A(\lambda) = \sum_{k\geq 0} A_k\lambda^{-k} \end{aligned} \tag{90}$$

Clearly, similar conclusion can be obtained for $\chi_3(y,x,\lambda)$ which by (79) and for the mn-th energy level is equal to:

$$\chi_{3,mn}(y,x,\lambda) = -(-1)^n \chi_{1,m}(x,\lambda) = -(-1)^n A(\lambda) \sin\left(m\pi\frac{x}{a}\right) \tag{91}$$

Therefore coming back to (78) we get:

$$\Psi^{as}_{mn}(x,y,\lambda) = 2iA(\lambda)\sin\left(n\pi\frac{y}{b}\right)\sin\left(m\pi\frac{x}{a}\right) \tag{92}$$

which again is the result got in the previous section.

The energy E is given however by the **finite** semiclassical series:

$$E = \frac{1}{2}p^2 + \frac{E_1}{\lambda^2} = \frac{1}{2}\left(\left(\frac{n\pi}{\lambda b}\right)^2 + \left(\frac{m\pi}{\lambda a}\right)^2\right), \qquad m,n = 1,2,... \tag{93}$$

Let us stress the following main differences between the previous non-periodic case and the bouncing mode one despite the fact that in both the cases the results obtained are the same.

1. The bouncing ball skeleton is represented in the phase space by a cylinder rather than by a closed torus;

2. contrary to the non-periodic cases only one skeleton is sufficient in the bouncing mode case to get the whole spectrum of the energy;

3. all terms of the semiclassical series expansion of GSWF exist (do not vanish) in the bouncing mode case, while only the zeroth one in the non-periodic one;

4. the semiclassical series for the energy contains two first terms in the bouncing mode case and only zeroth non-vanishing term in the non-periodic one; and

5. the JWKB approximation of the energy does **not** coincide with its global value.

However similarly to the non-periodic case the bouncing ball solution is also regular and exact.

5.3.2 Periodic skeletons different than the bouncing ball ones - singular SWF's

Consider now skeletons which periodic rays do not bounce between the sides of the rectangle. The simplest such a case the fifth one in Fig.7 is shown in Fig.9. As in the non-periodic case there are again four bundles in the corresponding skeleton but contrary to the case mentioned only two rays (of four of them) belonging to two different bundles can meet at each point of the rectangle if this point does not lie on SD's.

We have to note also that the skeleton associated with this on Fig.9 differs from it by the opposite directions of rays, i.e. possible energy levels we get for these skeletons must be degenerated.

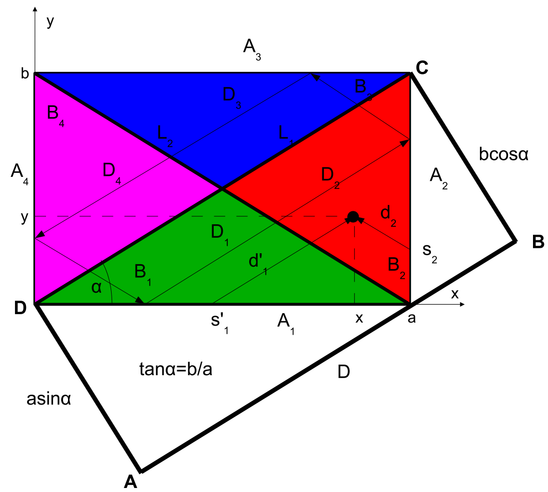


Figure 9: The singular skeleton defined by the corresponding pair L_1 and L_2 of SD's

The GSWF corresponding to the case looks as follows in different domains of the rectangles:

$$\Psi^{as}(x,y) = \begin{cases} e^{i\lambda p d_1 + i\lambda p s_1 \cos\alpha}\chi_1(d_1,s_1,\lambda) + e^{i\lambda p d_4' + i\lambda p s_4' \sin\alpha}\chi_4(d_4',s_4',\lambda) & (x,y)\in D_1 \\ e^{i\lambda p d_1' + i\lambda p s_1' \cos\alpha}\chi_1(d_1',s_1',\lambda) + e^{i\lambda p d_2 + i\lambda p s_2 \sin\alpha}\chi_2(d_2,s_2,\lambda) & (x,y)\in D_2 \\ e^{i\lambda p d_3 + i\lambda p s_3 \cos\alpha}\chi_3(d_3,s_3,\lambda) + e^{i\lambda p d_2' + i\lambda p s_2' \sin\alpha}\chi_2(d_2',s_2',\lambda) & (x,y)\in D_3 \\ e^{i\lambda p d_3' + i\lambda p s_3' \cos\alpha}\chi_3(d_3',s_3',\lambda) + e^{i\lambda p d_4 + i\lambda p s_4 \sin\alpha}\chi_4(d_4,s_4,\lambda) & (x,y)\in D_4 \\ \tan\alpha = \frac{b}{a} \end{cases} \tag{94}$$

where the variables s_k, s_k' are measured from the left ends of the corresponding sides $A_k,\ k = 1,...,4$.

The Dirichlet boundary conditions on the respective sides of the rectangle are therefore:

$$\begin{aligned} \chi_1(0,s,\lambda) + e^{i\lambda p b \sin\alpha}\chi_4\left(\frac{s}{\cos\alpha}, b - s\tan\alpha, \lambda\right) &= 0 \\ e^{i\lambda p a \cos\alpha}\chi_1\left(\frac{a-s}{\cos\alpha}, s, \lambda\right) + \chi_2(0,(a-s)\tan\alpha,\lambda) &= 0 \\ \chi_3(0,s,\lambda) + e^{i\lambda p b \sin\alpha}\chi_2\left(\frac{s}{\cos\alpha}, (a-s)\tan\alpha, \lambda\right) &= 0 \\ e^{i\lambda p a \cos\alpha}\chi_3\left(\frac{a-s}{\cos\alpha}, s, \lambda\right) + \chi_4(0, b - s\tan\alpha, \lambda) &= 0 \\ 0 < s < a & \end{aligned} \tag{95}$$

One can easily find from (95) that:

$$\chi_1(0,s,\lambda) = e^{2i\lambda p(b\sin\alpha + a\cos\alpha)}\chi_1\left(\frac{2a}{\cos\alpha}, s, \lambda\right) \tag{96}$$

or

$$e^{2i\lambda p(b\sin\alpha + a\cos\alpha)} = 1 \tag{97}$$

because $\chi_1(d,s,\lambda)$ is periodic with the period $\frac{2a}{\cos\alpha} = 2(b\sin\alpha + a\cos\alpha) = 2D$ where D is the length of the rectangle diagonal.

Therefore we get the following quantization condition for the zeroth energy term $E_0 = \frac{1}{2}p^2$

$$\lambda pD = n\pi, \qquad n = 1,2,... \tag{98}$$

Now we have to note that none of the bundle considered has a piece of its boundary common with any other one inside the rectangle. Seemingly as such could be considered the rectangle diagonals if the rays in the respective bundles were not run in the opposite directions. Therefore GSWF's defined in these bundles have to vanish on the respective diagonals of the rectangle so that we have to have:

$$\begin{aligned} \chi_k(d,0,\lambda) = 0, \qquad & k = 1,...,4 \\ & 0 \le d \le D \end{aligned} \tag{99}$$

But then from (95) we get also:

$$\chi_1(0,a,\lambda) = \chi_2(0,b,\lambda) = \chi_3(0,a,\lambda) = \chi_4(0,b,\lambda) = 0 \tag{100}$$

Further using the propagation formula (15) for $\chi_{1,1}(2D,s) = \chi_{1,1}(0,s)$ we get:

$$\chi''_{1,0}(s) + 2E_1\sin^2\alpha\chi_{1,0}(s) = 0 \tag{101}$$

which with the conditions (99)-(100) for $\chi_1(d,s,\lambda)$ gives:

$$\chi_{1,0}(s) = A_0\sin(\sqrt{2E_1}\sin\alpha s) \tag{102}$$

where E_1 defines the second term of the semiclassical energy expansion with the condition:

$$E_1 = \frac{1}{2}\frac{m^2\pi^2}{a^2\sin^2\alpha}, \qquad m = 1,2,3,... \tag{103}$$

Next repeating the procedure for the bouncing mode skeleton to the remaining terms $\chi_{1,k}(2D,s) = \chi_{1,k}(0,s)$, $k = 1,2,3,...$, we get for them:

$$\chi_{1,k}(s) = A_k\sin(\sqrt{2E_1}\sin\alpha s) \tag{104}$$

so that

$$\begin{aligned} \chi_1(d,s,\lambda) &= A(\lambda)\sin(\sqrt{2E_1}\sin\alpha s) \\ A(\lambda) &= \sum_{k\ge0} A_k\lambda^{-k} \end{aligned} \tag{105}$$

Therefore using (95) the final form of the SWF $\Psi^{as}(x,y)$ can be written as follows:

$$\Psi^{as}_{mn}(x,y) = \begin{cases} e^{i\lambda p_n(x\cos\alpha+y\sin\alpha)}\sin\left(m\pi\frac{1}{a}(x-y\cot\alpha)\right) - & \\ e^{i\lambda p_n(x\cos\alpha-y\sin\alpha)}\sin\left(m\pi\frac{1}{a}(x+y\cot\alpha)\right) & (x,y)\in D_1 \\ e^{i\lambda p_n(x\cos\alpha+y\sin\alpha)}\sin\left(m\pi\frac{1}{a}(x-y\cot\alpha)\right) + & \\ e^{i\lambda p_n(-x\cos\alpha+y\sin\alpha+2a\cos\alpha)}\sin\left(m\pi\frac{1}{a}(x+y\cot\alpha)\right) & (x,y)\in D_2 \\ -e^{i\lambda p_n(-x\cos\alpha-y\sin\alpha)}\sin\left(m\pi\frac{1}{a}(x-y\cot\alpha)\right) + & \\ e^{i\lambda p_n(-x\cos\alpha+y\sin\alpha+2a\cos\alpha)}\sin\left(m\pi\frac{1}{a}(x+y\cot\alpha)\right) & (x,y)\in D_3 \\ -e^{i\lambda p_n(-x\cos\alpha-y\sin\alpha)}\sin\left(m\pi\frac{1}{a}(x-y\cot\alpha)\right) - & \\ e^{i\lambda p_n(x\cos\alpha-y\sin\alpha)}\sin\left(m\pi\frac{1}{a}(x+y\cot\alpha)\right) & (x,y)\in D_4 \end{cases} \tag{106}$$

while the energy spectrum is:

$$E = \frac{1}{2}p^2 + \frac{E_1}{\lambda^2} = \frac{\pi^2}{2\lambda^2}\left(\frac{n^2}{D^2} + \frac{m^2D^2}{a^2b^2}\right), \qquad m, n = 1, 2, 3, ... \tag{107}$$

By their construction the solutions (106) are all singular - their derivatives are discontinuous on the rectangle diagonals. The corresponding energy spectrum also differs from the exact one as well as from the one of the regular SWF's obtained in the last two sections.

It should be stressed however that the solutions (106) - (107) are **allowed** semiclassical solutions to the rectangle billiards eigenvalue problem, which corresponds physically to effects of the short wave limits. It is therefore of great importance whether one can detect in these limits resonant modes in the respective rectangular cavity corresponding to the SWF's (106) and the energy spectrum (107). Indeed such modes have been detected experimentally by Bogomolny *et al* [18] for the rectangle cavity with a barrier inside. This case of the billiards will be discussed in the next sections.

If however such modes can be detected in the rectangle cavity then one can expect the corresponding GSWF's to have forms of standing waves rather than of the running ones as in (106). We can get such forms of GSWF's noticing that the spectrum (107) is obviously degenerate since in the case considered the associated skeleton $\mathbf{B}^T$ is different from $\mathbf{B}$. We can use therefore the corresponding running solutions for the skeleton $\mathbf{B}^T$ to construct by superpositions the standing SWF's corresponding to the energy spectrum (107). The simplest two superpositions are:

$$\Psi^{as}_{1,2;mn}(x,y) = \begin{cases} \sin\left(m\pi\frac{1}{a}(x - y\cot\alpha)\right) \times \begin{cases} \sin(\lambda p_n(x\cos\alpha + y\sin\alpha)) \\ \pm\cos(\lambda p_n(x\cos\alpha + y\sin\alpha)) \end{cases} - \\ \sin\left(m\pi\frac{1}{a}(x + y\cot\alpha)\right) \times \begin{cases} \sin(\lambda p_n(x\cos\alpha - y\sin\alpha)) \\ \cos(\lambda p_n(x\cos\alpha - y\sin\alpha)) \end{cases} & (x,y) \in D_1,\ D_4 \\ \\ \sin\left(m\pi\frac{1}{a}(x - y\cot\alpha)\right) \times \begin{cases} \sin(\lambda p_n(x\cos\alpha + y\sin\alpha)) \\ \pm\cos(\lambda p_n(x\cos\alpha + y\sin\alpha)) \end{cases} - \\ \sin\left(m\pi\frac{1}{a}(x + y\cot\alpha)\right) \times \begin{cases} \sin(\lambda p_n(x\cos\alpha - y\sin\alpha - 2a\cos\alpha)) \\ -\cos(\lambda p_n(x\cos\alpha - y\sin\alpha - 2a\cos\alpha)) \end{cases} & (x,y) \in D_2,\ D_3 \end{cases} \tag{108}$$

where the plus corresponds to the domains D_1, D_2 and the minus - to D_3, D_4.

The above results on the singular SWF's can be generalized to arbitrary periodic skeletons in the rectangular billiards some of which are shown on Fig.6 and Fig.7. This can be done by noticing that an arbitrary periodic SD is defined by arbitrary two relatively prime numbers $\{p, q\}$ so that a SD in a rectangle with the sides a and b shown in Fig.9 starting from the vertex $(0, 0)$ and being inclined by an angle α to the x-axis is defined by such two numbers as follows:

$$\tan\alpha = \frac{pb}{qa} \tag{109}$$

The above fact follows directly from the unfolded forms of periodic trajectories shown in Fig.6 and Fig.7 if one realizes that each of them has to finish on another vertex of the

rectangle. It follows also that the set of all SD is countable but dense among all trajectories in the rectangle.

Pairs $\{p,q\}$ can appear in the following combinations $\{e,o\}$, $\{o,o\}$ and $\{o,e\}$ where e stands for "even" and o - for "odd". The respective SD's defined by these combinations finish their runs in the vertices $(a,0)$, (a,b) and $(0,b)$ correspondingly.

A SD defined by a pair $\{p,q\}$ bounces $p-1$-times from each horizontal side of the rectangle and $q-1$-times - from each of the vertical ones. If D denotes its global length measured from its starting vertex $(0,0)$ to one of its final ones just enumerated then $D = pb\sin\alpha + qa\cos\alpha = \sqrt{(qa)^2+(pb)^2}$.

If a SD is chosen, i.e. $\{p,q\}$ are fixed, and it ends at one of the vertices just enumerated then the second SD which has to accompany the chosen one to built the skeleton starts and ends at the remaining two of these vertices. Note that a number of bundles in such a skeleton is then equal to $2p+2q$ while their widths are equal to $\frac{a}{p}\sin\alpha = \frac{b}{q}\cos\alpha$.

The quantization formula (98) remains then valid for the case considered while (103) takes the form $E_1 = \frac{1}{2}\frac{m^2p^2\pi^2}{a^2\sin^2\alpha}$, $m = 1,2,3,...$, so that the formula (107) for the energy spectrum remains also unchanged. This formula corresponds to the spectrum of the exact standing wave functions in a rectangle with the sides $D \times \frac{a}{p}\sin\alpha$. According to Bogomolny and Schmit [17] this rectangle can be considered as an unfolded skeleton so that each such a standing wave function defined on it should generate a corresponding semiclassical one defined on the skeleton by folding appropriately the rectangle mentioned to the real skeleton and interfering pieces of the standing wave function in crossed points of such a folding. However such a procedure to be correct still needs for the resulting SWF's to vanish on the rectangular billiards sides. It is seen that such a procedure though theoretically possible and correct is complicated enough to be replaced by the corresponding constructions of the SWF's in the real folded skeleton according to the rules formulated in sec.3.

Nevertheless the SWF's corresponding to the energy spectrum (107) are all singular having discontinuous derivatives on the lines separating two neighboring bundles. The lines are just the bundles boundaries on which the running SWF's defined in these bundles have to vanish.

5.3.3 Regular SWF's in the periodic skeletons in rectangular billiards

The singular SWF's found in the previous section provides us with the energy spectrum which is different from the regular one. However for a particular relations between the rectangle sides these solutions can become regular. This can happen if one assumes the following two additional conditions:

$$\begin{aligned} \lambda pb\sin\alpha &= k\pi \\ \lambda pa\cos\alpha &= l\pi \\ k,l &= 1,2,.... \end{aligned} \tag{110}$$

where p is the global momentum of the billiards ball.

While it is tedious to be checked the singular SWF's satisfying the conditions (110) become then regular obtaining the following form:

$$\tilde{\Psi}^{as}_{1,2;mn}(x,y) = \sin\left(\frac{m-l}{a}p\pi x\right)\sin\left(\frac{m+k}{b}q\pi y\right) \pm \sin\left(\frac{m+l}{a}p\pi x\right)\sin\left(\frac{m-k}{b}q\pi y\right) \tag{111}$$

valid in the whole rectangle area.

An immediate consequence of the conditions (110) is that they limit the form of the rectangles for which the solution (111) can exist. Namely we have to have:

$$\tan^2\alpha = \left(\frac{pb}{qa}\right)^2 = \frac{k}{l} \qquad k,l = 1,2,.... \tag{112}$$

Let us make further an important note that despite its semiclassical origin the solution (111) is still exact being a linear combination of two solutions given by the formula (50). But since it is an eigenfunction of a given energy this combination means that both its terms have to be eigenfunctions of the same energy, i.e. we should have:

$$\frac{(m-l)^2p^2}{a^2} + \frac{(m+k)^2q^2}{b^2} = \frac{(m+l)^2p^2}{a^2} + \frac{(m-k)^2q^2}{b^2} \tag{113}$$

It is easy to check that this is the case if one takes into account the condition $(pb)^2l = (qa)^2k$ which follows from (112).

Therefore if $k_0,\ l_0$ for which α satisfies (112) are relatively prime then the remaining allowed pairs of $k,\ l$ satisfying the condition (112) are of course of the form $k = nk_0,\ l = nl_0,\ n = 1,2,...$, and the energy spectrum formula (107) for such a rectangle takes the following final form:

$$E = \frac{\pi^2}{2\lambda^2}\frac{(m^2+n^2k_0l_0)(k_0+l_0)p^2}{a^2k_0}, \qquad m,n = 1,2,3,... \tag{114}$$

Each energy level (114) is degenerate with the following base in their two dimensional degeneracy space:

$$\begin{aligned}\Phi_{1,mn}(x,y) &= \sin\left(\frac{m-nl_0}{a}p\pi x\right)\sin\left(\frac{m+nk_0}{b}q\pi y\right)\\ \Phi_{2,mn}(x,y) &= \sin\left(\frac{m+nl_0}{a}p\pi x\right)\sin\left(\frac{m-nk_0}{b}q\pi y\right)\\ &\qquad\qquad m,n = 1,2,3,...\end{aligned} \tag{115}$$

To conclude for the "periodic" skeleton considered in this section to get regular SWF's it is necessary for a rectangle to satisfy first the constrain (110). But if it happens then the corresponding SWF's coincide with the ones built on "generic" skeletons considered in sec. 4.1. An additional result of the considered periodic skeleton configurations is that the corresponding energy levels are degenerate.

Let us stress however that if there are no such four natural numbers k,l,p,q by which for a given rectangle the condition (112) can be satisfied then none periodic orbit skeleton provides us with a possibility to construct on it regular SWF's except the bouncing ball skeletons. For these two cases the energy spectra obtained coincides with the one got from the "generic" skeleton calculations and their degeneracy disappears.

5.4 Quantization of pseudointegrable systems - broken rectangles

By a broken rectangle we mean the one which can be decomposed into a finite set of disjoint rectangles, see Fig.10. If reintegrated it shows some number of rectangular bays and peninsulas.

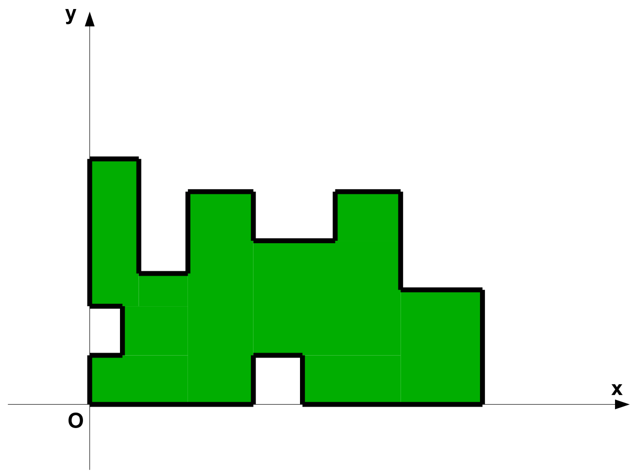


Figure 10: An "arbitrary" broken rectangular billiards

In fact the broken rectangles can serve as archetypes of pseudointegrable systems with an arbitrary genus. Since however we are interested in considering some special SWF's configurations related to classical periodic trajectories we shall limit ourselves to rather simple forms of the broken rectangles. The simplest one with a single peninsula (the L-shaped pseudointegrable billiards in terms of Kudrolli and Sridhar [2]) is shown in Fig.11 and also in Fig.12 and Fig.13 with several skeleton configurations related to some SD.

5.4.1 Regular SWF's in the broken rectangular billiards

Skeletons in the broken rectangular billiards which are to provide us with the SWF's which would be the exact solutions to the corresponding eigenvalue problem have to be regular. Considering the billiards of Fig.11 it is clearly seen that there are no such skeletons - each "generic" skeleton sketched on Fig.11 has to have the bundles, denoted by 5. and 6. in the figure, which can be composed with the bundles 2. and 4. respectively into two global but singular compound bundles. Therefore each "generic" skeleton has to be singular. Despite this the Lagrange surface form by these skeletons are closed and of genus 2.

The first general conclusion which follows for the considered case of the broken rectangular billiards and the more so for the more complicated ones of Fig.10 is that one cannot expect the obtained SWF's to be exact.

To convince oneself of the correctness of the last conclusion we will consider the bouncing ball modes skeleton of Fig.12C,D, instead of making a tedious calculations for the generic skeletons leading however to the same results. To this goal it is enough to match the corresponding GSWF's defined on the skeletons shown in Fig.12C,D according to the conditions (32). Both the GSWF's have the form (92). Therefore the respective procedure leads us to the following quantization conditions for the energy E_{nm}:

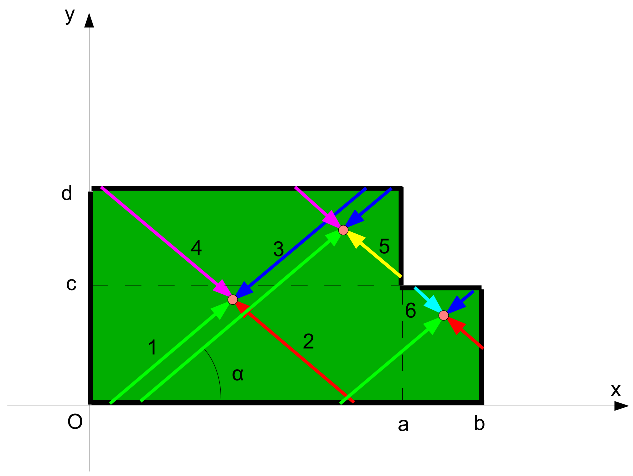


Figure 11: A single bay rectangular billiards with a "generic" global singular skeleton composed of six bundles. Its reduced form contains four global but singular compound bundles which form in the phase space the closed Lagrange surface of genus 2.

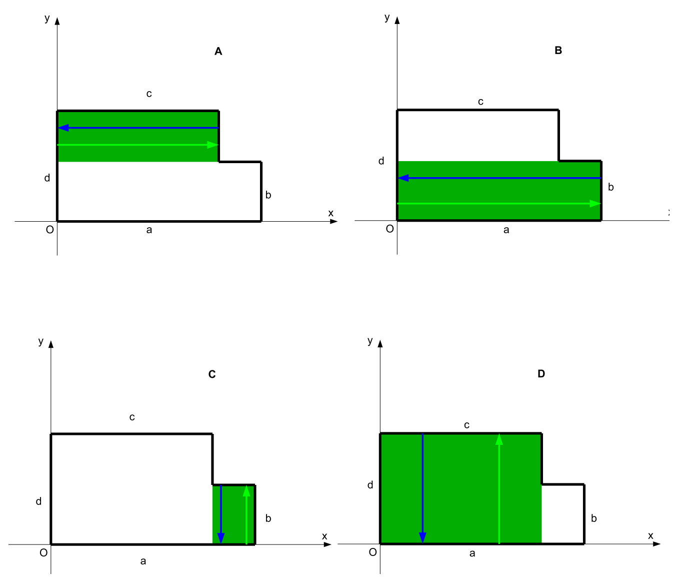


Figure 12: A single bay rectangular billiards with singular skeletons built on periodic trajectories - the singular bouncing ball cases. Every skeleton forms the cylinder-like Lagrange surface in the phase space.

$$\frac{a}{n_0} = \frac{c}{l_0} = n\frac{\Lambda_x}{2}, \qquad \frac{d}{m_0} = \frac{b}{k_0} = m\frac{\Lambda_y}{2}$$
$$\Lambda_x = \frac{2\pi}{\sqrt{2E_1}}, \qquad \Lambda_y = \frac{2\pi}{\lambda p}$$
$$E_{nm} = \frac{2\pi^2}{\lambda^2}\left(\frac{1}{\Lambda_x^2} + \frac{1}{\Lambda_y^2}\right) = \frac{\pi^2}{2\lambda^2}\left(\frac{n^2 n_0^2}{a^2} + \frac{m^2 m_0^2}{d^2}\right)$$
$$m, n = 1, 2, ... \tag{116}$$

where $\Lambda_x,\ \Lambda_y$ are the wave lengths of rays in the horizontal and vertical skeletons respectively shown in Fig.12 and n_0, m_0 are the smallest integers satisfying $l_0 a = n_0 c$ and $k_0 d = m_0 b$ where l_0, k_0 are also integers.

The respective SWF's are the following:

$$\Psi_{nm}^{as}(x,y,\lambda) = \begin{cases} A\sin\frac{2\pi x}{\Lambda_x}\sin\frac{2\pi y}{\Lambda_y} = A\sin(\frac{nn_0}{a}\pi x)\sin\left(\frac{mm_0}{d}y\right) & (x,y) \in D_{br} \\ 0 & (x,y) \notin D_{br} \end{cases} \tag{117}$$

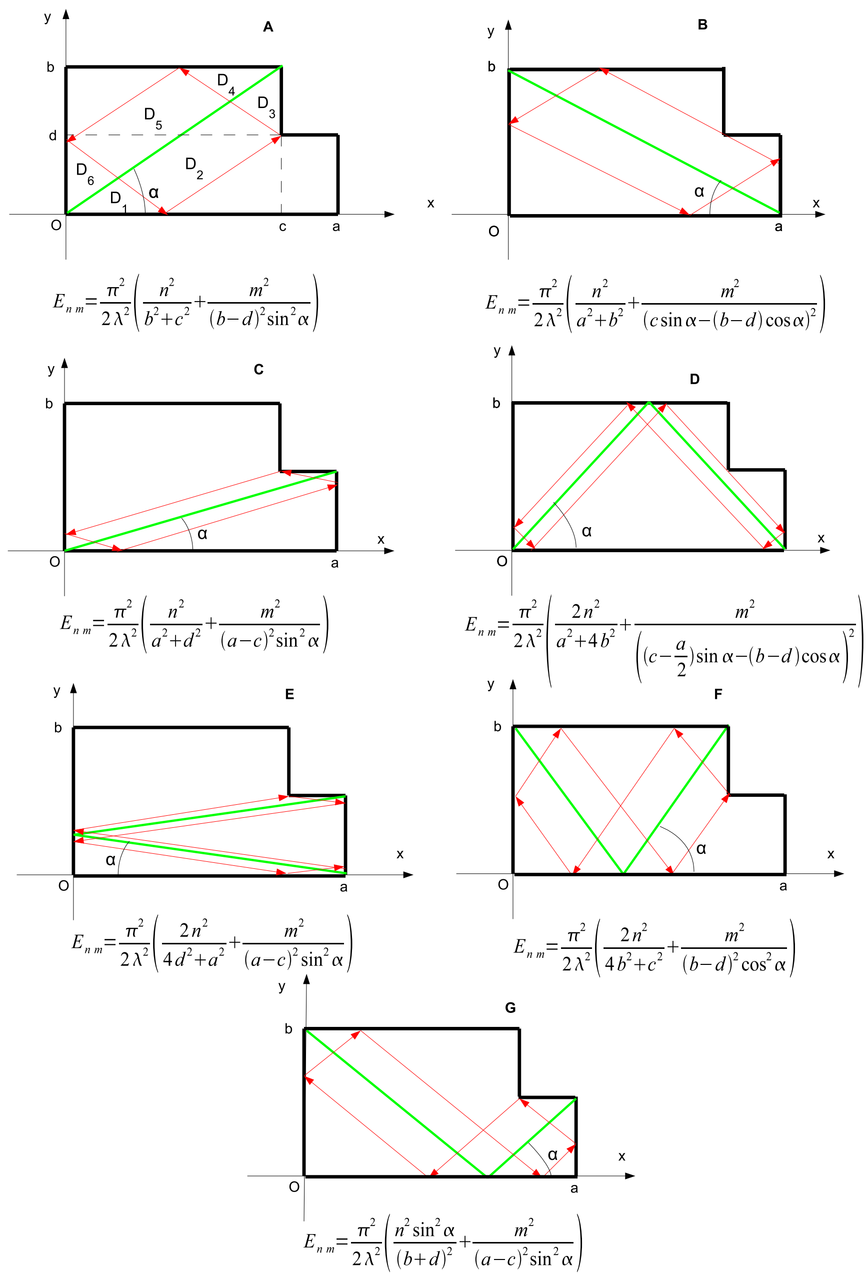
where D_{br} denotes the domain of the x,y-plane occupied by the broken rectangular of Fig.11.

The above GSWF is of course regular. Nevertheless due to the properties of the bouncing ball skeleton it is not exact despite the fact that it satisfies the Dirichlet boundary conditions as well as the SE. The reasons for its approximate character are the conditions (116) which cannot be satisfied if the corresponding length a, b, c, d of the broken rectangle are not commensurate by pairs. However if such a commensurability is satisfied by the sides of the broken rectangle then the solution (117) is exact.

One can also easily realize that the last results can be generalized to any bouncing ball modes skeleton in the broken rectangular billiards of Fig.10. A little bit surprising is that the semiclassical formulae (116) for the energy and (117) for the wave functions remain unchanged for any such a billiards while a number of conditions the wave lengths Λ_x and Λ_y have to satisfy filling the vertical and horizontal skeletons by integer numbers of their halves is increasing respectively to numbers of bays and peninsulas forming the sides od such billiards. In fact for the corresponding SWF's the half wave lengths $\frac{1}{2}\Lambda_x$ and $\frac{1}{2}\Lambda_y$ considered as the units of lengths on the respective horizontal and vertical sides of the broken rectangle have to measure these sides by integers. It means of course that these sides have to be commensurate so theoretically such a condition excludes SWF's for most the broken rectangular billiards. Practically however since incommensurability in fact does not exists in real measurements by experimental errors one can always tune the corresponding waves to the real dimensions of the broken rectangles.

5.4.2 Singular SWF's in the broken rectangular billiards

The SWF's (117) seem to be the unique regular ones which can be constructed in the broken rectangular billiards, i.e. any other SWF's should be singular. Examples of them corresponds to all the skeletons shown in Fig.12 and Fig.13. The skeletons of Fig.12 define singular SWF's which are identical with the ones of the formulae (116) and (117) except that there are no relations between the rectangular sides a, b, c, d. These modes were observed experimentally by Kudrolli and Sridhar [2].



A: $$E_{nm}=\frac{\pi^2}{2\lambda^2}\left(\frac{n^2}{b^2+c^2}+\frac{m^2}{(b-d)^2\sin^2\alpha}\right)$$

B: $$E_{nm}=\frac{\pi^2}{2\lambda^2}\left(\frac{n^2}{a^2+b^2}+\frac{m^2}{(c\sin\alpha-(b-d)\cos\alpha)^2}\right)$$

C: $$E_{nm}=\frac{\pi^2}{2\lambda^2}\left(\frac{n^2}{a^2+d^2}+\frac{m^2}{(a-c)^2\sin^2\alpha}\right)$$

D: $$E_{nm}=\frac{\pi^2}{2\lambda^2}\left(\frac{2n^2}{a^2+4b^2}+\frac{m^2}{\left((c-\frac{a}{2})\sin\alpha-(b-d)\cos\alpha\right)^2}\right)$$

E: $$E_{nm}=\frac{\pi^2}{2\lambda^2}\left(\frac{2n^2}{4d^2+a^2}+\frac{m^2}{(a-c)^2\sin^2\alpha}\right)$$

F: $$E_{nm}=\frac{\pi^2}{2\lambda^2}\left(\frac{2n^2}{4b^2+c^2}+\frac{m^2}{(b-d)^2\cos^2\alpha}\right)$$

G: $$E_{nm}=\frac{\pi^2}{2\lambda^2}\left(\frac{n^2\sin^2\alpha}{(b+d)^2}+\frac{m^2}{(a-c)^2\sin^2\alpha}\right)$$

Figure 13: A single bay rectangular billiards with singular skeletons built on the shortest periodic trajectories different than the bouncing ball ones with the energy spectra of the respective GSWF's. Note that $n, m = 1, 2, ...,$ for every spectrum. The red arrows show the outermost periodic orbits of the corresponding skeletons. The skeletons form the cylinder-like Lagrange surface in the phase space each.

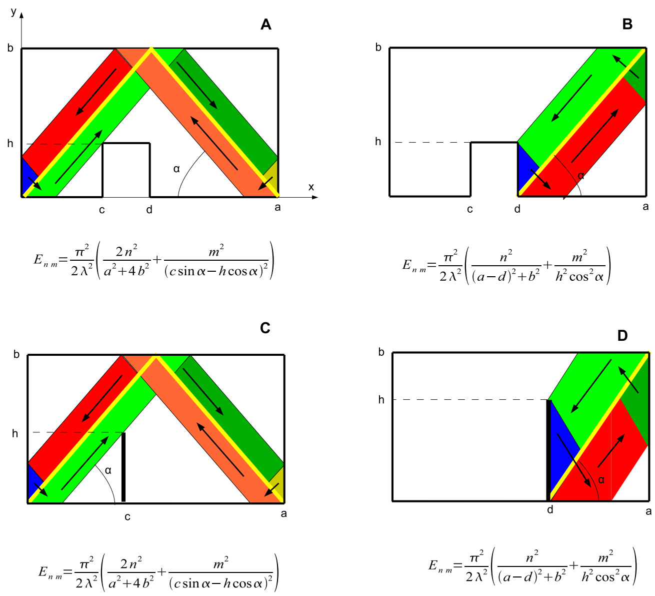


Figure 14: A broken rectangular billiards with barriers and with possible superscar skeletons and with the energy spectra corresponding to the respective GSWF's. In all the above formulae $n, m = 1, 2, ...$ The corresponding Lagrange surfaces in the phase space are cylinder-like.

Even more spectacular are skeletons built by periodic orbits different than the bouncing ball ones shown in Fig.13. These are just the skeletons which provide us with the singular SWF's with properties described by Bogomolny and Schmit [17] as superscars and was observed also experimentally by Kudrolli and Sridhar [2].

The singular SWF's corresponding to the broken rectangle billiards shown in Fig.14 (upper figures) were observed by Bogomolny *et al* [18]. In fact the authors mentioned considered the limit of the billiards when $d - c \to 0$ (lower figures). They studied experimentally the high frequency modes in a microwave cavity [18] confirming the existence of the superscar modes predicted earlier by Bogomolny and Schmit [17].

For a completeness we shall give below the form of the singular SWF's for the skeleton A of Fig.13 together with their degenerate energy spectrum.

$$\Psi^{as;sing}_{1,2;mn}(x,y) =$$

$$\begin{cases} \sin\left(\frac{m\pi}{c-d\cot\alpha}(x-y\cot\alpha)\right) \times \begin{cases} \sin(\frac{n\pi}{D}(x\cos\alpha+y\sin\alpha)) \\ \pm\cos(\frac{n\pi}{D}(x\cos\alpha+y\sin\alpha)) \end{cases} - \\ \sin\left(\frac{(m\pi}{c-d\cot\alpha}(x+y\cot\alpha)\right) \times \begin{cases} \sin(\frac{n\pi}{D}(x\cos\alpha-y\sin\alpha)) \\ \cos(\frac{n\pi}{D}(x\cos\alpha-y\sin\alpha)) \end{cases} \\ (x,y)\in D_1,\ D_6 \\ \sin\left(\frac{m\pi}{c-d\cot\alpha}(x-y\cot\alpha)\right) \times \begin{cases} \sin(\frac{n\pi}{D}(x\cos\alpha+y\sin\alpha)) \\ \pm\cos(\frac{n\pi}{D}(x\cos\alpha+y\sin\alpha)) \end{cases} \\ (x,y)\in D_2,\ D_5 \end{cases}$$

$$
\begin{cases}
\sin\left(\frac{m\pi}{c-d\cot\alpha}(x-y\cot\alpha)\right)\times\begin{cases}\sin(\frac{n\pi}{D}(x\cos\alpha+y\sin\alpha))\\ \pm\cos(\frac{n\pi}{D}(x\cos\alpha+y\sin\alpha))\end{cases} - \\
\sin\left(\frac{m\pi}{c-d\cot\alpha}(x+y\cot\alpha)\right)\times\begin{cases}\sin(\frac{n\pi}{D}(x\cos\alpha-y\sin\alpha-2c\cos\alpha))\\ -\cos(\frac{n\pi}{D}(x\cos\alpha-y\sin\alpha-2c\cos\alpha))\end{cases}\\
(x,y)\in D_3,\ D_4
\end{cases}
$$

$$
\tan\alpha = \frac{b}{c}
$$

$$
\lambda p D = n\pi
$$

$$
E_1 = \frac{m^2\pi^2}{2(c\sin\alpha - d\cos\alpha)^2}
$$

$$
E_{mn} = \frac{1}{2}p^2 + \frac{E_1}{\lambda^2} = \frac{\pi^2}{2\lambda^2}\left(\frac{n^2}{D^2} + \frac{m^2}{(c\sin\alpha - d\cos\alpha)^2}\right), \qquad m,n = 1,2,3,... \tag{118}
$$

where $D = \sqrt{b^2+c^2}$ is the length of the diagonal shown in the Fig.13A.

5.5 The rational polygon billiards other than the rectangular ones - the triangle and the pentagon billiards

In this and in the next section we will made a short review of the billiards systems which have been widely [8, 10, 18, 19, 20] considered both theoretically and experimentally having mainly in mind their skeleton description.

5.5.1 The equilateral triangle billiards

These billiards which dynamics is integrable have been considered in past very often [8, 23]. From the skeleton construction view point one can similarly to the rectangular case built the "generic" skeleton, see Fig.15a, as well as skeleton generated by periodic orbits, Fig.15b-d.

In the "generic" case of Fig.15a the corresponding skeletons consist of twenty four bundles. The reduced skeletons however have them already twelve and such a number counts a set of the BSWF's which as it is seen from Fig.15a have to interfere to built the GSWF. The skeleton is regular and the result of such a superposition is well known [8, 23] so we do not perform a corresponding calculations which leads to the exact eigenfunctions and the energy levels as well of the corresponding eigenvalue problems.

It has to be stressed however that since the associated skeletons $\mathbf{B}^A$ coincide with $\mathbf{B}$ then the energy level degeneracy if happens has to be of different origins.

Nevertheless we will report here the corresponding results for the periodic skeletons of Fig.15b-c. The skeleton of Fig.15b is defined by the SD composed of the three sides of the triangle so that the period of the orbit is equal to three. The skeleton contains only three bundles covering the whole triangle each so it is regular. To get GSWF it is enough to superpose in each point of the triangle only three BSWF defined on the bundles. It is obvious also that for the skeleton shown in Fig.15b its associated one does not coincide with it having the same energy, i.e. the corresponding energy spectrum is degenerate. Taking therefore two independent superpositions of their GSWF we get the following regular semiclassical solutions for the case:

$$
\Psi_{mn}^{(1)}(x,y) = \sin\left(\frac{2}{3}m\pi x\right)\sin\left(\frac{2}{\sqrt{3}}n\pi y\right) - \sin\left[\frac{1}{3}m\pi(x+\sqrt{3}y)\right]\sin\left[n\pi\left(x-\frac{y}{\sqrt{3}}\right)\right] +
$$

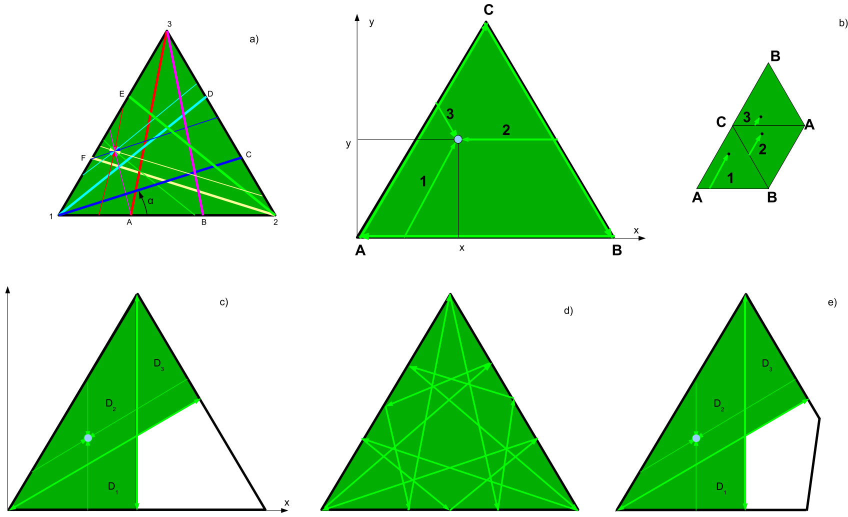


Figure 15: Skeletons in the equilateral triangles: a) - a "generic" regular one forming a torus in the phase space, b) - defined by periodic orbits (thick arrows) forming a Möbius band in the phase space, c) - defined by periodic orbits with a cylinder-like Langrange surface, d) - with longer periodic orbits, e) - in the amputated corner triangle. The latter can be neither integrable nor pseudointegrable.

$$\sin\left[\frac{1}{3}m\pi(x-\sqrt{3}y)\right]\sin\left[n\pi\left(x+\frac{y}{\sqrt{3}}\right)\right] \quad (119)$$

and

$$\Psi_{mn}^{(2)}(x,y)=\cos\left(\frac{2}{3}m\pi x\right)\sin\left(\frac{2}{\sqrt{3}}n\pi y\right)+\cos\left[\frac{1}{3}m\pi(x+\sqrt{3}y)\right]\sin\left[n\pi\left(x-\frac{y}{\sqrt{3}}\right)\right]-$$
$$\cos\left[\frac{1}{3}m\pi(x-\sqrt{3}y)\right]\sin\left[n\pi\left(x+\frac{y}{\sqrt{3}}\right)\right] \quad (120)$$

with the energy spectrum:

$$E_{mn}=\frac{2\pi^2}{9\lambda^2}(m^2+3n^2)$$
$$m,n=1,2,... \quad (121)$$

for both the solutions where m,n are even or odd simultaneously.

Note that since both the above solutions are regular they are exact as well as their common energy spectra.

In the case of the periodic skeleton of Fig.15c defined by the periodic orbit of the length $\sqrt{3}$ (the double height of the triangle) there are four bundles of the skeletons and the corresponding four BSWF's which are singular since the skeleton is singular. Therefore the GSWF is defined locally in the domains $D_k,\ k=1,2,3,$ of Fig.16c being however continuous in $D_1\cup D_2\cup D_3$. It is the following:

$$\Psi_{mn}^{as}(x,y)=$$

$$\begin{cases} \sin\left(2m\pi\frac{x}{\sqrt{3}}\right)\sin(2n\pi x) & (x,y)\in D_1 \\ \sin\left(2m\pi\frac{x}{\sqrt{3}}\right)\sin(2n\pi x)-\sin\left[m\pi\left(x+\frac{y}{\sqrt{3}}\right)\right]\sin\left[n\pi\left(x+\sqrt{3}y\right)\right] & (x,y)\in D_2 \\ -\sin\left[m\pi\left(x+\frac{y}{\sqrt{3}}\right)\right]\sin\left[n\pi\left(x+\sqrt{3}y\right)\right] & (x,y)\in D_3 \end{cases} \tag{122}$$

with the energy spectrum:

$$E_{mn}=\frac{2\pi^2}{3\lambda^2}(m^2+3n^2)$$
$$m,n=1,2,... \tag{123}$$

Of course $\Psi^{as}_{mn}(x,y)\equiv 0$ if $(x,y)\notin D_1\cup D_2\cup D_3$.

While the above $\Psi^{as}_{mn}(x,y)$ satisfies Schrödinger equation its derivatives are not continuous on the boundaries separating the domains $D_k,\ k=1,2,3$. This is why it is only semiclassical approximation to the exact solutions given earlier.

The spectrum (123) are naturally degenerate. However a source of this degeneracy is the symmetry of the equilateral triangle. Namely, there are solutions with the spectrum (123) which can be obtained by rewriting the solution (122) in the new coordinate systems obtained from the present one by moving it to the remaining two vertices and rotating it by $\pm\frac{2\pi}{3}$ respectively.

These new solutions can of course interfere with (122) and with themselves so that trying to stimulate the corresponding state in a cavity certainly such superposed states will be generated rather than the "pure" state (122).

To isolate however the state (122) it is enough to remove one of the three corners of the triangle as it is shown in Fig.15e.

The skeleton corresponding to the periodic orbit shown in Fig.15d contains eighteen bundles none pair of which can be done a compound one. The length of the orbit is equal to $D=3\sqrt{7}$ while the wideness of each bundle is equal to $w=\frac{1}{2}\sqrt{\frac{3}{7}}$. Therefore the GSWF's for this case are singular and are linear combinations of the eighteen BSWF's. The energy spectrum is degenerate at least because the skeleton shown and its associated do not coincide. Its form can be obtained using the following general formula for the energy spectrum of the periodic skeletons:

$$E_{mn}=\frac{1}{2}p^2+\frac{E_1}{\lambda^2}=\frac{1}{2}\frac{4m^2\pi^2}{\lambda^2D^2}+\frac{n^2\pi^2}{2\lambda^2w^2}=\frac{\pi^2}{\lambda^2}\left(\frac{2m^2}{D^2}+\frac{n^2}{2w^2}\right) \tag{124}$$

where D is the length of the orbit period and w is the wideness of the skeleton stripe.

Therefore for the case considered we get:

$$E_{mn}=\frac{2\pi^2}{63\lambda^2}(m^2+21n^2)$$
$$m,n=1,2,... \tag{125}$$

with some possible relations between m and n which can follow from the corresponding possible relations between χ-factors of the BSWF's defined on the skeleton bundles and which can be established by detailed constructions of the corresponding GSWF's.

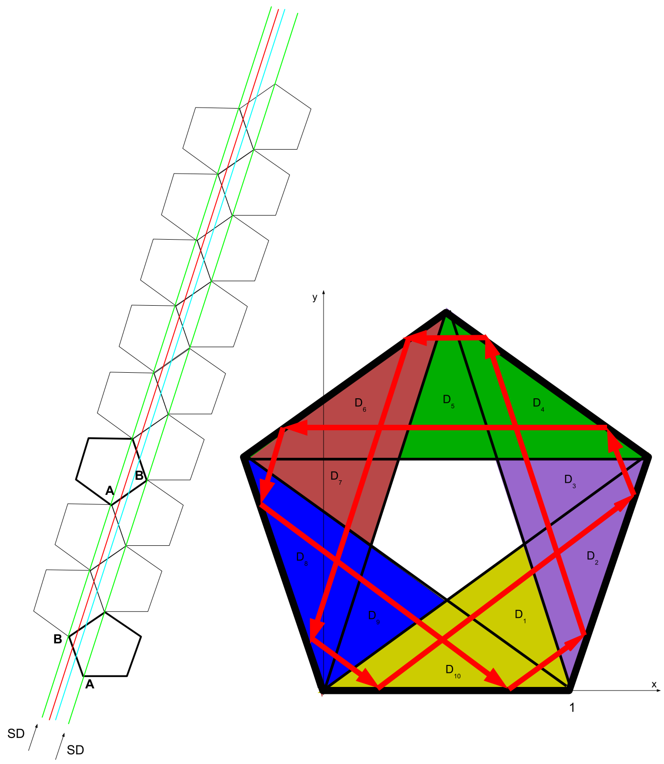


Figure 16: The pentagon billiards and its simplest whispering gallery superscar skeleton with five bundles. A typical periodic orbit is shown (the red straight line in the unfolded pentagon) the limiting form of which is the five-pointed star orbit (the SD green straight lines). It degenerates into the inscribed pentagon orbit (the blue straight line) with the half of the typical period equal to $10\cos\frac{\pi}{5} \simeq 8.090$. Gluing the end segments indicated by AB in the unfolded pentagon we get a Möbius band in the phase space.

5.5.2 The pentagon billiards

The pentagon form billiards were also the subject of intensive studies of both theoretical and experimental [19, 20]. In the latter case the corresponding pentagon cavities were made of some dielectric media. In the very high frequency region the corresponding electromagnetic waves form different modes among which the whispering gallery one of Fig.16 was the most prominent. The other pentagon modes shown in the paper of Lebental *et al* [19] are more difficult for an identification in terms of the corresponding skeletons also because of different boundary conditions the authors wanted to consider.

Nevertheless in our paper we would like to distinguish other pentagon modes of SWF's. We will not however consider in details a "generic" mode because of its complexity. Namely in such a case one can find simply by hand that there are twenty compound bundles composing the corresponding skeletons and consequently the same number of BSWF's which have to interfere to get the GSWF's satisfying the Dirichlet boundary conditions say. These compound bundles however are regular each so the corresponding skeletons are regular and

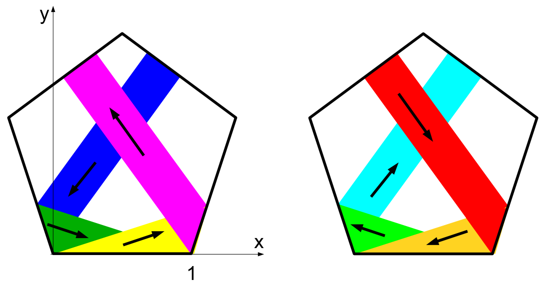


Figure 17: Another simple singular periodic skeleton in the pentagon billiards with the half of the period equal to $3\cos\frac{\pi}{10}+2\sin\frac{\pi}{5}\simeq 4.029$. Its Lagrange surface is cylinder-like.

the GSWF's built on such skeletons will be regular and by that will be exact solutions to the corresponding SE with the exact energy spectrum despite the fact that these results can be obtained by the semiclassical approach. Therefore instead of the "generic" skeleton cases we consider the singular ones generated by periodic orbits.

The simplest of such cases is the whispering gallery skeleton shown in Fig.16. It is quite easy to write the corresponding singular GSWF. Nevertheless we limit ourselves to quote merely the corresponding result for the energy spectrum. Namely we have:

$$E_{mn}=\frac{\pi^2}{2\lambda^2}\left(\frac{m^2}{25\cos^2\frac{\pi}{5}}+\frac{n^2}{\sin^2\frac{\pi}{5}}\right)$$
$$m,n=1,2,... \tag{126}$$

where similarly to the triangle case the number m,n are simultaneously even or odd.

The spectrum (126) is of course degenerate. Note also that in the pentagon (white) center the corresponding GSWF's vanish identically.

Another case of the singular skeleton shown in Fig.17 reminds the rectangular bouncing ball modes and the similar triangle modes of Fig.15. The corresponding energy spectrum is:

$$E_{mn}=\frac{\pi^2}{2\lambda^2}\left(\frac{m^2}{(3\cos\frac{\pi}{10}+2\sin\frac{\pi}{5})^2}+\frac{n^2}{\sin^2\frac{\pi}{10}}\right)$$
$$m,n=1,2,... \tag{127}$$

This spectrum is of course degenerate - there are five independent solutions with the same spectrum. In the pentagon cavity all these solutions can be stimulated simultaneously. To isolate at least one of them it is enough to desymmetrize the pentagon into its forms shown for examples in Fig.18. For the case a) of the figure the corresponding energy spectrum is:

$$E_{mn}=\frac{\pi^2}{2\lambda^2}\left(\frac{m^2}{(2\cot\frac{\pi}{10}+\cos\frac{\pi}{10})^2}+\frac{n^2}{\sin^2\frac{\pi}{10}}\right)$$
$$m,n=1,2,... \tag{128}$$

while for the case b) it is:

$$E_{mn}=\frac{\pi^2}{2\lambda^2}\left(\frac{m^2}{(2\tan\frac{\pi}{10}+3\cos\frac{\pi}{10})^2}+\frac{n^2}{\sin^2\frac{\pi}{10}}\right)$$

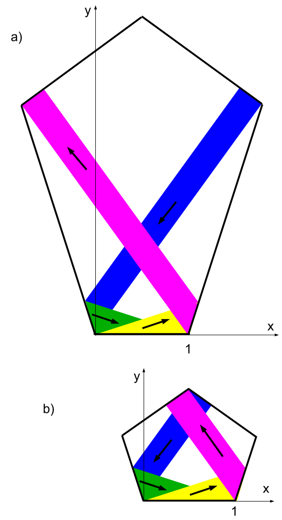


Figure 18: The deformed pentagon billiards with the singular periodic skeletons having the half periods equal to $2\cot\frac{\pi}{10}+\cos\frac{\pi}{10}\simeq 7.106$ - the case $a)$ and $2\tan\frac{\pi}{10}+3\cos\frac{\pi}{10}\simeq 3.503$ - the case $b)$

$$m,n=1,2,... \tag{129}$$

6 Billiards with chaotic classical motions - superscars and periodic orbits

6.1 Singular SWF's in the chaotic polygon based billiards

It follows from the previous section that the idea of the skeletons seems to be effective in solving some simple situations of quantum phenomena related semiclassically with billiards which shapes stimulate rather chaotic than regular (integrable or pseudointegrable) motions. An example of such cases is shown in Fig.15e. Still more spectacular situations exist in billiards which can be obtained from the rectangular and the pentagonal ones by their deformations. Examples of such deformations and the superscar modes possible to be detected in such chaotic billiards are shown in Fig.18 and Fig.19. To describe analytically the superscar skeletons shown in these figures the methods of the previous sections can be applied directly. Note that the superscar mode corresponding to the Sinai billiards of Fig.19 was observed experimentally by Kudrolli and Sridhar [2] and by Sridhar and Heller [21] who studied the Sinai billiards also numerically (see also [5]).

7 Summary and conclusions

In this paper we have formulated very thoroughly the idea of skeletons in billiards and we have shown in great details how to construct on them semiclassical wave functions basing on the modified Maslov - Fedoriuk approach [14]. The modification mentioned utilizes rather

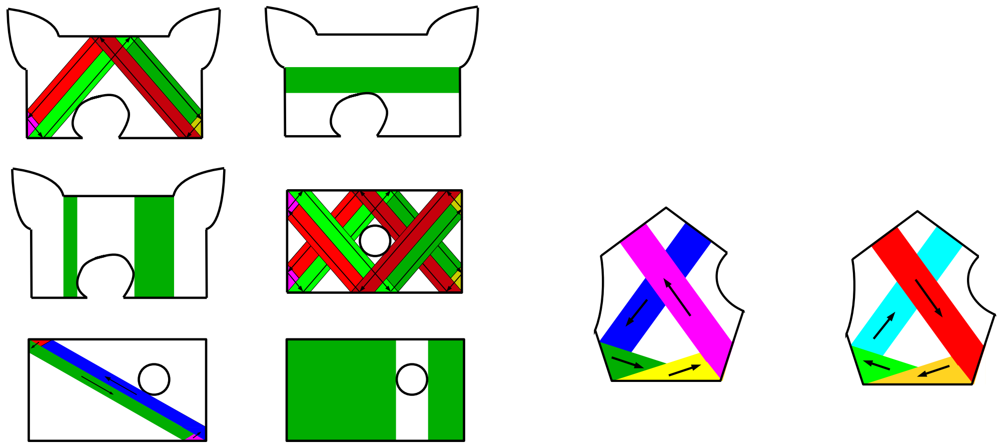


Figure 19: The deformed rectangular, the Sinai and the pentagon billiards giving rise to chaotic motions with some possible superscar skeletons

the corresponding configuration spaces than the phasespaces making use of the complex time to continue SWF's by caustics [22].

In general the skeletons in billiards play a role of the Arnold tori [15] in the integrable dynamical systems with smooth motions.

In this paper we have considered mainly the rational polygon billiards beginning with the rectangular one to show that:

- a huge variety of skeletons can be found in such billiards;
- by the Keller - Rubinov [23] and the Ritchens - Berry [8] constructions global skeletons form in the phase space the closed Lagrangian surfaces of some genus (≥ 1);
- the singular skeletons which are not global are represented in the phase space by open surfaces with boundaries. Most of the periodic skeletons belong to such a class and form in the phase space the cylinder-like or the Möbius-like bands;
- the SWF's which solve the energy eigenvalue problems in such billiards can be of two kinds - the regular and the singular ones - depending on the skeletons which they are constructed on;
- the regular SWF's solve the energy eigenvalue problems exactly;
- the singular SWF's solve the energy eigenvalue problems only approximately providing us with the superscar solutions of Bogomolny and Schmit [1, 2, 3, 17, 18, 19, 20];
- the superscar skeletons and SWF's can be found in many chaotic billiards which contain flat boundaries in favourable patterns [2, 3, 5, 21].

Some general conclusion which can be done by reassuming our results is that in addition to the saturation of the Gutzwiller formula [24] for the semiclassical Green functions the other role of the periodic orbits is their being of a local organizer of order in the integrable as well as in the pseudointegrable and the chaotic motions realized by the respective skeletons.

Appendix A

In this appendix we are going to show, that the geometrical optics rule of reflections of rays off the billiards boundary is a consequence of demands of vanishing on the boundary of the linear combination (28) accompanied by the conditions (29) and (30). Namely consider the following superposition of SWF's:

$$\begin{aligned} \Psi_k^{as}(x,y;u,l;\lambda) = \Psi_{k,1}^{\sigma_1}(d_1,s_1;u,l;\lambda) + \Psi_{k,2}^{\sigma_2}(d_2,s_2;u,l;\lambda) = \\ J_{k,1}^{-\frac{1}{2}}(d_1,s_1;u,l)e^{i\sigma_1 k\left(d_1+\int_0^{s_1}\cos\alpha_{k,1}(s';u,l)ds'\right)}\chi_{k,1}^{\sigma_1}(d_1,s_1;u,l;\lambda) + \\ J_{k,2}^{-\frac{1}{2}}(d_2,s_2;u,l)e^{i\sigma_2 k\left(d_2+\int_0^{s_2}\cos\alpha_{k,2}(s';u,l)ds'\right)}\chi_{k,2}^{\sigma_2}(d_2,s_2;u,l;\lambda) \end{aligned} \tag{130}$$

with

$$\begin{aligned} \mathbf{r} \equiv [x,y] = \mathbf{r}_{k,1}(d_1,s_1;u,l) = \mathbf{r}_0(s_1) + \mathbf{d}_1(s_1;u,l) = \\ \mathbf{r}_{k,2}(d_2,s_2;u,l) = \mathbf{r}_0(s_2) + \mathbf{d}_2(s_2;u,l) \\ \mathbf{r}_{k,1}(d,s;u,l) \in B_k(u,l), \qquad \mathbf{r}_{k,2}(d,s;u,l) \in B'_k(u,l), \qquad B_k(u,l) \neq B'_k(u,l) \end{aligned} \tag{131}$$

i.e. the SWF's $\Psi_{k,1}^{\sigma_1}(d_1,s_1;u,l;\lambda)$ and $\Psi_{k,2}^{\sigma_2}(d_2,s_2;u,l;\lambda)$ are defined respectively on the bundles $B_k(u,l)$ and $B'_k(u,l)$ with $D_k(u,l) \cap D'_k(u,l) \neq \oslash$ interfering in the crossing point $[x,y]$ of two rays $\mathbf{r}_{k,1}(d_1,s_1;u,l)$ and $\mathbf{r}_{k,2}(d_2,s_2;u,l)$ belonging to the respective bundles.

Therefore the condition for $\Psi_k^{as}(x,y,\lambda)$ to vanish on $A_k(u,l)$ is:

$$\begin{aligned} J_{k,1}^{-\frac{1}{2}}(0,s)e^{ik\sigma_1\int_0^s\cos\alpha_{k,1}(s';u,l)ds'}\chi_{k,1}^{\sigma_1}(s;u,l;\lambda) + \\ J_{k,2}^{-\frac{1}{2}}(0,s)e^{ik\sigma_2\int_0^s\cos\alpha_{k,2}(s';u,l)ds'}\chi_{k,2}^{\sigma_2}(s;u,l;\lambda) = 0 \\ \mathbf{r}_0(s) \in A_k(u,l) \end{aligned} \tag{132}$$

Because of the k-dependence the last relation can be satisfied if and only if:

$$\sigma_1\int_0^s \cos\alpha_{k,1}(s';u,l)ds' = \sigma_2\int_0^s \cos\alpha_{k,2}(s';u,l)ds', \qquad \mathbf{r}_0(s) \in A_k(u,l) \tag{133}$$

It is easy to see however that there are only two solutions of the last condition:

$$\begin{aligned} \alpha_{k,1}(s;u,l) \equiv \alpha_{k,2}(s;u,l) \qquad for \qquad \sigma_1 = \sigma_2 \\ \alpha_{k,1}(s;u,l) \equiv \pi - \alpha_{k,2}(s;u,l) \qquad for \qquad \sigma_1 = -\sigma_2 \\ \mathbf{r}_0(s) \in A_k(u,l) \end{aligned} \tag{134}$$

The first solutions are however uninteresting identifying the bundles in a given segment and consequently leading to the solutions vanishing identically on $A_k(u,l)$.

Putting $\alpha_{k,1}(s;u,l) \equiv \alpha(s;u,l)$ and $\sigma_1 = -\sigma$ we get from the second solution and from (132):

$$\begin{aligned} \chi_{k,2}^{-\sigma}(s;u,l;\lambda) = -\chi_{k,1}^{\sigma}(s;u,l;\lambda) \equiv \chi_k(s;u,l;\lambda) \\ \mathbf{r}_0(s) \in A_k(u,l) \end{aligned} \tag{135}$$

so that the combination (28) becomes:

$$
\begin{aligned}
\Psi_k^{as}(x,y;u,l;\lambda) = \Psi_{k,1}^{\sigma}(d_1,s_1;u,l;\lambda) - \Psi_{k,2}^{-\sigma}(d_2,s_2;u,l;\lambda) = \\
J_{k,1}^{-\frac{1}{2}}(d_1,s_1;u,l)e^{i\sigma k\left(d_1+\int_0^{s_1}\cos\alpha(s';u,l)ds'\right)}\chi_{k,1}^{\sigma}(d_1,s_1;u,l;\lambda) - \\
J_{k,2}^{-\frac{1}{2}}(d_2,s_2;u,l)e^{-i\sigma k\left(d_2+\int_0^{s_2}\cos\alpha(s';u,l)ds'\right)}\chi_{k,2}^{-\sigma}(d_2,s_2;u,l;\lambda) \\
\mathbf{r}_0(s) \in A_k(u,l)
\end{aligned}
\tag{136}
$$

where $\chi_k\sigma(d,s;u,l;\lambda)$, $\sigma=\pm$, are given by (27) with $\chi_k^{\sigma}(0,s;u,l;\lambda) \equiv \chi_k(s;u,l;\lambda)$.

The last result shows that $\Psi_k^{as}(x,y;u,l;\lambda)$ vanishing on $A_k(u,l)$ has to be represented semiclassically by a combination of at least two SWF's of opposite signatures and such that if $\Psi_{k,1}^{\sigma}(d,s;u,l;\lambda)$ is defined on the bundle $B_k(u,l)$ then the second SWF $\Psi_{k,2}^{-\sigma}(d,s;u,l;\lambda)$ has to be defined on the bundle $B_k^A(u,l)$.

Appendix B

It is shown in this appendix that $\delta_k(u_j,l_j)$ from the formula (38) is s-independent. To this end consider Fig.1 on which a mapping $h_k(s;u,l)$ of the arc $A_k(u,l)$ into an arc $A_{k'}(u',l')$ is defined by the the bundle $B_k(u,l)$. According to this mapping we have:

$$
\begin{aligned}
\mathbf{r}_0(h_k(s;u,l)) = \mathbf{r}_0(s) + \mathbf{D}(s;u,l) \\
\mathbf{r}_0(s) \in A_k(u,l), \qquad \mathbf{r}_0(h_k(s;u,l)) \in A_{k'}(u',l')
\end{aligned}
\tag{137}
$$

where $\mathbf{D}(s;u,l)$ is a vector linking the point $\mathbf{r}_0(s)$ with the point $\mathbf{r}_0(h_k(s;u,l))$ of the billiards boundary.

Differentiating the equation (137) with respect to s we get:

$$
\begin{aligned}
\frac{\partial h_k(s;u,l)}{\partial s}\cos\beta(h_k(s;u,l)) = \cos\beta(s) - D(s;u,l)\frac{\partial\gamma_k(s;u,l)}{\partial s}\sin\gamma_k(s;u,l) + \\
\cos\gamma_k(s;u,l)\frac{\partial D(s;u,l)}{\partial s} \\
\frac{\partial h_k(s;u,l)}{\partial s}\sin\beta(h_k(s;u,l)) = \sin\beta(s) + D(s;u,l)\frac{\partial\gamma_k(s;u,l)}{\partial s}\cos\gamma_k(s;u,l) + \\
\sin\gamma_k(s;u,l)\frac{\partial D(s;u,l)}{\partial s}
\end{aligned}
\tag{138}
$$

where $D(s;u,l)$ is the length of $\mathbf{D}(s;u,l)$.

Making further the proper linear combinations of the last equations we have finally:

$$
\begin{aligned}
\frac{\partial h_k(s;u,l)}{\partial s}\cos\alpha_{k'}(h_k(s;u,l);u',l') = \cos\alpha_k(s;u,l) + \frac{\partial D(s;u,l)}{\partial s} \\
-\frac{\partial h_k(s;u,l)}{\partial s}\sin\alpha_{k'}(h_k(s;u,l);u',l') = \sin\alpha_k(s;u,l) - D(s;u,l)\frac{\partial\gamma_k(s;u,l)}{\partial s}
\end{aligned}
\tag{139}
$$

where we have taken into account the following relation between the angles involved:

$$
\begin{aligned}
\alpha_{k'}(h_k(s;u,l);u',l') + \alpha_k(s;u,l) = \beta(h_k(s;u,l)) - \beta(s) + 2\pi \\
\gamma_k(s;u,l) = \beta(s) + \alpha_k(s;u,l)
\end{aligned}
\tag{140}
$$

which follows from Fig.1.

The independence of s of $\delta_k(u_j,l_j)$ follows now easily from the first of the relations (139).

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